Week 6: Phase Estimation and the RSA Cryptosystem

COMS 4281 (Fall 2024)

Admin

1. Practice worksheet out, and quiz #3 will be out tonight.
2. Midterm on October 21. More details soon.

Last time

- Discrete Fourier Transform F_N is a **unitary matrix** mapping standard basis $\{|0\rangle, \ldots, |N-1\rangle\}$ to Fourier basis $\{|f_0\rangle, |f_1\rangle, \ldots, |f_{N-1}\rangle\}$ where

$$|f_j\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{\frac{2\pi ijk}{N}} |k\rangle \ .$$

- The Quantum Fourier Transform is a fast **quantum algorithm** that implements the DFT F_N for $N = 2^n$, and runs in time $\mathrm{poly}(n) = \mathrm{poly}(\log N)$.

Brief linear algebra review

Eigenvalues

If $M \in \mathbb{C}^{N \times N}$ is a matrix, $|\psi\rangle \in \mathbb{C}^N$ is a vector, and $\lambda \in \mathbb{C}$ satisfying

$$M |\psi\rangle = \lambda |\psi\rangle$$

then we say that $|\psi\rangle$ is an **eigenvector** of M with **eigenvalue** λ.

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On the other hand,

$$(\lambda^* \langle\psi|)(\lambda|\psi\rangle) = (\langle\psi| U^\dagger)(U|\psi\rangle) = \langle\psi| U^\dagger U |\psi\rangle = \langle\psi|\psi\rangle = 1$$

because $U^\dagger U = I$ (one of definitions of being unitary).

Eigenvalues of unitary matrices

Fact: The eigenvalues of a unitary matrix U are all of the form $e^{2\pi i\theta}$ for some $\theta \in [0, 2\pi)$.

Proof continued: Therefore

$$|\lambda|^2 = 1$$

and the only such λ's possible are of the form $e^{2\pi i\theta}$.

Some examples

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$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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We see that

$$Z|0\rangle = |0\rangle \qquad Z = |1\rangle = -|1\rangle \ .$$

Therefore standard basis are the eigenvectors and ± 1 are corresponding eigenvalues.

Some examples

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We can compute this by hand, or we can also remember that

$$X |+\rangle = |+\rangle \qquad X |-\rangle = - |-\rangle$$

so the Hadamard basis are the eigenvectors and ± 1 are the corresponding eigenvalues.

Some examples

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1. $|0,0\rangle$ with eigenvalue 1
2. $|0,1\rangle$ with eigenvalue 1
3. $|1,+\rangle$ with eigenvalue 1
4. $|1,-\rangle$ with eigenvalue -1

Phase Estimation Algorithm

Application of QFT: Phase Estimation

Phase Estimation Algorithm (PEA) is one of the most important subroutines in quantum computing.

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Goal of PEA:

- Ability to run controlled versions of U^k for $k = 1, 2, \ldots$.
- An **eigenstate** $|\psi\rangle$ where $U|\psi\rangle = e^{2\pi i\theta}|\psi\rangle$,

estimate θ.

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Answer: It becomes a **relative** phase once you run the controlled-U gate in superposition:

$$\begin{aligned} cU\left|+\right\rangle\left|\psi\right\rangle &= \frac{1}{\sqrt{2}}(\left|0\right\rangle\left|\psi\right\rangle + \left|1\right\rangle U\left|\psi\right\rangle) \\ &= \frac{1}{\sqrt{2}}(\left|0\right\rangle\left|\psi\right\rangle + e^{2\pi i\theta}\left|1\right\rangle\left|\psi\right\rangle) \\ &= \frac{1}{\sqrt{2}}(\left|0\right\rangle + e^{2\pi i\theta}\left|1\right\rangle)\left|\psi\right\rangle \end{aligned}$$

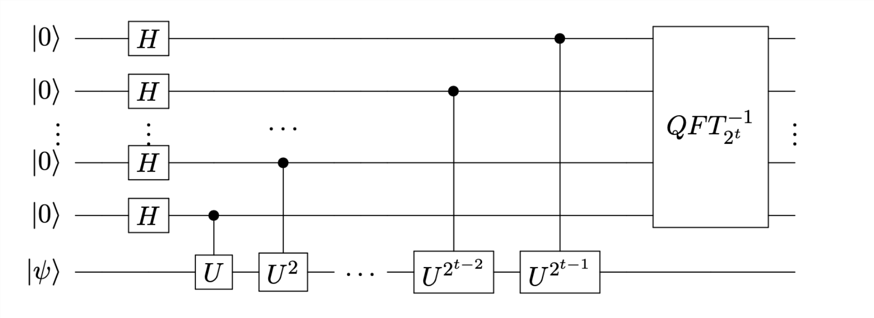
Phase Estimation Algorithm

Assume for simplicity that θ can be represented using exactly t bits. In other words the binary representation of θ looks like

$$\theta = 0.\theta_1\theta_2 \cdots \theta_t$$

where $\theta_1, \theta_2, \ldots \in \{0, 1\}$. This is equivalent to

$$\theta = \frac{\theta_1}{2} + \frac{\theta_2}{2^2} + \cdots + \frac{\theta_t}{2^t}.$$



Measuring the first t qubits will yield $|\theta_1, \theta_2, \ldots, \theta_t\rangle$.

Phase Estimation Algorithm Analysis

Let's analyze a special case where $t = 2$, and $\theta = \frac{\theta_1}{2} + \frac{\theta_2}{4}$ for $\theta_1, \theta_2 \in \{0, 1\}$.

(On the board...)

Phase Estimation Algorithm Analysis

Question: What if the phase θ cannot be exactly expressed as t bits?

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Answer: If we use $t + k$ ancilla qubits, and measure only the first t ancilla qubits, we will get the best t-bit approximation $\widetilde{\theta}$ of θ with probability $1 - 2^{-k}$.

Phase Estimation Algorithm Analysis

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Answer: The set $\{|\phi_j\rangle\}$ of eigenvectors of U forms a basis for $\mathbb{C}^{2^n}$ (if U is n-qubit unitary). We can write $|\psi\rangle$ as

$$|\psi\rangle = \sum_j \alpha_j |\phi_j\rangle$$

for some coefficients α_j.

Running Phase Estimation on $|\psi\rangle$ with ancilla qubits $|0 \cdots 0\rangle$ yields a state that is close to

$$\approx \sum_j \alpha_j |\phi_j\rangle \otimes |\widetilde{\theta}_j\rangle$$

where $\widetilde{\theta}_j$ is an approximation of the eigenphase θ_j, i.e. $U|\phi_j\rangle = e^{2\pi i \theta_j} |\phi_j\rangle$.

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Measuring the last register yields $\widetilde{\theta}_j$ with probability $|\alpha_j|^2$.

RSA and the Factoring problem

RSA Cryptosystem

- Invented by Rivest, Shamir, and Adleman in 1977
- Most widely deployed public-key cryptosystem
- Enables public-key encryption as well as digital signatures

Public key encryption

1. Bob generates a *secret-key/public-key* pair (sk, pk), and publishes pk on the internet.
2. Alice uses pk and her message m to create a *ciphertext* c which she sends to Bob.
3. Bob gets c, and uses sk to decode m.
4. The adversary sees (pk, c), and should get no information about m.

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Bob

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4. Set public key $pk = (e, N)$, and secret key $sk = d$.

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This works because $c^d = (m^e)^d = m^{ed}$, and modulo N this equals m by *Fermat's Little Theorem*.

RSA Cryptosystem

Adversary sees the public key $pk = (e, N)$ and the encrypted message (ciphertext) c.

It does not know the primes p, q, nor the secret key $sk = d$.

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If it knew the prime factorization $N = pq$ it could compute the secret key!

Factoring problem

Input: Positive integer N.

Output: Prime factorization of N as $p_1^{a_1} p_2^{a_2} \cdots$.

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The prime factorization of N is unique by the **Fundamental Theorem of Arithmetic**.

To find a factorization of N, it suffices to be able to find *some* nontrivial divisor of N.

Factoring problem

It is widely believed that Factoring is hard for classical computers.
The best classical algorithm, known as the **General Number Field Sieve**, takes time roughly

$$\exp\left(O(\log N)^{1/3}\right) .$$

This is essentially **exponential** in the number of digits of N.

Shor's algorithm

A quantum algorithm to solve Factoring in $\mathrm{poly}(\log N)$ steps.

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Shor's Algorithm is also a hybrid classical-quantum algorithm.

1. **Classical part**: reduce the factoring problem to **order finding**.
2. **Quantum part**: solve order finding.

Order Finding

Input: given positive integers N, x such that

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2. $\gcd(N, x) = 1$ (i.e. they do not have any nontrivial factors in common)

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Output: find smallest integer r such that $x^r = 1 \bmod N$ (called the **order** of $x \bmod N$).

Next time

A quantum algorithm to solve Order Finding