AdS/CFT Correspondence and Quantum Error Correction

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1 Introduction

The AdS/CFT corresponence is an exciting area of research that relates conformal field theories to phenomena in bulk AdS space. One example of such a relation is a quantum error correcting code. In this writeup, we will explain the basics of AdS/CFT correspondence and the connection to quantum error correction. We will present an example of a quantum error correcting code proposed by John Preskill, Daniel Harlow, Beni Yoshida, and Fernando Pastawski that is motivated by the AdS/CFT correspondence. This writeup will explore the AdS/CFT correspondence mainly using language suited to quantum information rather than quantum gravity in the interest of clarity and concision. Finally, we will briefly discuss some open problems.

2 AdS/CFT

The AdS/CFT correspondence is a conjectured mapping between states and operators in a n-dimensional anti-deSitter space and states and operators in a $(n-1)$-dimensional conformal field theory. The hyperbolic anti-deSitter space is called the bulk, and the conformal field theory is called the boundary. The "correspondence" between the bulk and the boundary is a mapping of operators and states known as the "dictionary." The dictionary is currently incomplete, meaning that we do not know how to map every state/operator in the bulk to the boundary, or vice versa, in every dimension. Over the past several years, AdS/CFT correspondence has been one of the most promising methods of investigating quantum gravity and quantum field theories. [Mal99]

2.1 AdS: Anti-deSitter Space

2.1.1 Physical Interpretation

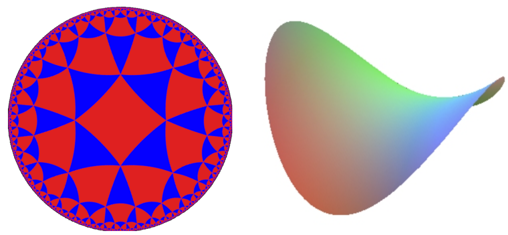
Anti-deSitter Space is a specific solution to Einstein's Field Equations that uses a negative cosmological constant. There are two other solutions that are often discussed: deSitter Space and Minkowski space, which use a positive and zero cosmological constant, respectively. The cosmological constant corresponds to the energy density of the vacuum in the solution, so a negative cosmological constant corresponds to negative vacuum energy density, a zero cosmological constant to zero vacuum energy density, and a positive cosmological constant to positive vacuum energy density.

2.1.2 Mathematical Interpretation

Mathematically, anti-deSitter space is a manifold that has some useful properties. Specifically, it is maximally symmetric and has constant negative curvature. When a space is maximally symmetric, it is both *homogeneous* and *isotropic*. We can also say an N-dimensional space M is maximally symmetric if and only if it has $\frac{N(N+1)}{2}$ Killing Vectors. Intuitively, this means that a maximally symmetric space looks the same from any location and from any angle. See Appendix 1 for more details. The anti-deSitter space is negatively curved, and the curvature is the same everywhere.

2.1.3 Visualization

There are two useful visualizations of a uniformly negatively curved space. The first is one that we will see again in our discussion of quantum error correcting codes, the uniform hyperbolic tiling, in which all of the red regions have the same area and all of the blue regions have the same area:



Images thanks to [Pas+15] and [FER]

The image on the right is a negatively curved space in three dimensions. The two dimensional case, on the left, has curvature in one fewer dimensions. Intuitively, one can picture a person walking from the center to the boundary in a straight line as seen from the top. The person will appear to slow as they approach the boundary because the direction of travel is no longer orthogonal to your "viewing angle."

2.1.4 Other Details

One important feature of AdS, for our purposes, is that due to the negative curvature, parallel timelike geodesics will eventually intersect. Another important thing to note is that we do not actually live in AdS. Most experimental evidence points to a positive vacuum energy density, which means we live in deSitter space. This AdS formulation, although it does not describe the larger geometry of the universe, has been useful in describing certain other physical phenomena and systems at small scales.

2.2 Conformal Field Theories

A conformal field theory is a type of quantum field theory that is invariant under conformal transformations.

2.2.1 Quantum Field Theory

Quantum field theory combines classical field theory, quantum mechanics, and special relativity. The uncertainty principle from quantum mechanics demonstrates that energy fluctuates greatly within a short period of time, and special relativity demonstrates that energy and mass are interchangeable. Therefore, Quantum field theory is a many body theory that describes the creation and annihilation of particles [Zee03]. While a classical field can be thought of as a function from a point in space time to a real or complex number, quantum field is a function from a point in space time to an operator in the Hilbert space. For example, the quantum field for a free spin-0 particle can be expressed as:

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2w_k}(a_k e^{ikx} + a_k^\dagger e^{-ikx})$$

where x is the position of particle in 4-dimensional space time, k is the momentum, $a_k^\dagger$ is the creation operator, and a_k is the annihilation operator [Sre07].

2.2.2 Conformal transformation

A conformal transformation $x \to x'$ rescales the metric by $g_{\alpha\beta}(x) \to \Omega^2(x) g_{\alpha\beta}(x)$. Conformal transformation includes rotation or Lorentz transformation, coordinate translation, and scale transformation. Note that conformal transformation preserves angles. [Ton12]

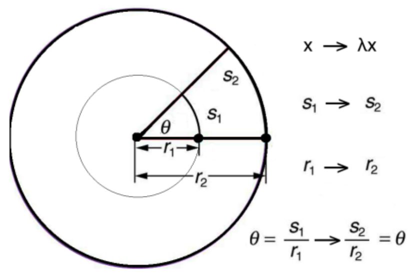


Figure 1: Conformal Transformation preserves angles [Qua16]

2.2.3 Other Details

CFT has important application in string theory and statistical mechanics. In string theory, strings trace out world sheets on which CFT determines the equation of motion of strings. In statistical mechanics, CFT offers a description of the system at critical points, where the correlation length approaches infinity and scale invariant theory becomes suitable. Many (1+1)-dimensional CFTs are solvable, but higher dimensional CFTs are generally much harder to study.

2.3 (2+1) AdS and (1+1) CFT: Our Favorite Case

For the purposes of motivating a quantum error correcting code, we will consider 2+1 dimensional AdS and 1+1 dimensional CFTs. (Spatial dimansions + time dimension) Our bulk-boundary space is constructed as follows. The bulk is constructed of a continuous stack of hyperbolic disks, so it looks like a cylinder. Each disk is further displaced in the time dimension than the one underneath it, so we can consider slices or discs further up in the cylinder to be further ahead in time. The CFT describes the behavior of space-time at the boundary of this cylinder, and the claim is that we have a dictionary to map operators in the bulk to operators on the boundary and vice versa using Rindler Wedge Reconstruction [ADH15] (Section 2).

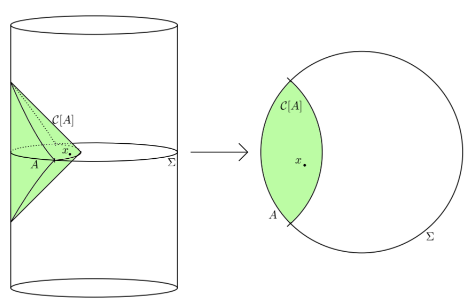


Figure 2: The operator x can be reconstructed on any part of the boundary surface A. For our purposes, we will be using a 2-dimensional tensor network, so we can say that any operator x within the green "causal wedge" can be reconstructed on the sub-region A of the circumference of the bulk.[Pas+15]

Additionally, when considering a uniform tessellation of the AdS bulk, we can apply the Ryu-Takyanagi Formula:

$$S_A = \frac{\text{Area}(\gamma_A)}{4G} \tag{1}$$

This equation is saying that for particular kinds of CFTs, the entropy S_A of a boundary sub-region A is equivalent to the area of minimal surface connecting the boundary of A through the bulk divided by a constant. (G is Newton's constant). In this case, the entropy describes to what degree the boundary region A is entangled with the rest of the boundary. This formula is important in motivating the connection between AdS/CFT and Tensor Networks.

3 Quantum Error Correction

3.1 Quantum Errors

Quantum systems are susceptible to errors due to interactions between qubits and environment. When an isolated quantum system interacts with the environment, it experiences *decoherence*–the quantum system becomes entangled with the environment, and the entanglement within the quantum system breaks down. When a qubit is completely mixed with the environment, it reaches a maximally mixed state and is considered to be "erased" from the quantum system. This is called an *erasure error*. Quantum Error correction aims to protect qubits from decoherence by encoding a logical qubit into an entangled quantum state composed of multiple physical qubits.

3.2 From Classical Error Correction to Quantum Error Correction

In classical computation, bit flip errors occurs with small probability, and they are corrected using the idea of redundant encoding, where we encode one logical bit into multiple physical bits, and recover the logical bit by taking the physical bit that has maximum occurrence in the logical bit. For example, if we encode 0 by 000 and one error occurs, 100, 010, or 001 would allow us to recover 0.

Comparing to classical error correction, quantum error correction has three challenges. Firstly, non-cloning theorem states that quantum states cannot be copied, so redundant encoding cannot be directly applied. Secondly, any measurement destroys superposition of quantum states. Finally, in addition to discrete bit flip errors, continuous errors like phase shift by certain angles exist for quantum states. In fact, these challenges can be overcome, and certain errors can be corrected through quantum error correction codes (QECC). QECC defines a map from k logical qubits to n physical qubits.

3.3 Stabilizer Codes

Stabilizer formalism provides a way to characterize the types of errors a code can correct. A stabilizer S of a code is an abelian subgroup of the n-qubit Pauli group $\pm\{I, X, Y, Z\}^{\otimes n}$. The *stabilizer code* is defined to be a subspace $\mathcal{H}_s \subseteq \mathcal{H}_{2^n}$ that is the simultaneous eigenspace of all elements of S with eigenvalue one. If S has $n-m$ generator $M_1, ..., M_{n-m}$, then $\mathcal{H}_s$ has dimension 2^m. When acting on states in $\mathcal{H}_{2^n}$, the generators have eigenvalue ± 1. In the code subspace $\mathcal{H}_s$, generators has eigenvalue $+1$. Thus, the stabilizer can detect error syndromes by checking the "parity" of different parts of a state. A stabilizer code that encodes n physical qubits to k logical qubits with distance d between two code word states can be denoted as a $[[n, k, d]]$ code. A code with distance $d = 2t + 1$ can correct t errors. A detailed description of a $[[5, 1, 3]]$ stabilizer code will be shown in section 5.1.

3.4 Concatenated codes

A concatenated code is a code with additional levels of encoding, and concatenation allows us to increase the number of errors a QECC can correct. For example, to concatenate a $[[n_1, 1, d_1]]$ code C_1 with a $[[n_2, 1, d_2]]$ code C_2, we further encode the a logical qubit of C_1 to each physical qubit of C_2. The resulting concatenated code has $n = n_1n_2, k = 1, d \geq d_1d_2$. In general, with concatenation, the number of required physical qubits and the number of errors that can be corrected grows exponentially.

4 Motivating QECC with AdS/CFT

4.1 Perfect Tensors

Perfect Tensors are a class of tensors defined in [Pas+15] that are necessary for the construction of the tensor networks (See 4.2) that will ultimately constitute our error correcting code. Perfect tensors are a special class of isometric tensors.

[Perfect Tensor] A $2n$ index tensor $T_{a_1,a_2,...,a_{2n}}$ is a perfect tensor if for any bipartition of its indices into two sets, A and A^C with $|A| \leq |A^C|$, T is proportional to an isometric tensor from A to A^C. [Pas+15] (Section 2)

4.1.1 Applications

These perfect tensors can be used to represent pure states. [Pas+15] Specifically, a pure state $|\psi\rangle$ with m v-dimensional spins can be written

$$|\psi\rangle = \sum_{a_1,a_2,...,a_m} T_{a_1,a_2,...,a_m} |a_1, a_2, ..., a_m\rangle \tag{2}$$

where T is a tensor with m indices, each with v degrees of freedom. Additionally, a perfect tensor with $2n$ indices represents a state with $2n$ qubits (assuming each index has two degrees of freedom). Any set of n qubits is maximally entangled with the complementary set, and there is an isometry between any two groups of indices. One can see how a bipartition into one qubit and $2n-1$ qubits could potentially motivate an error correcting code.

4.2 Tensor Networks

Representing m-spin system can take a large amount of computation memory because the required memory grows exponentially with m. This has posed a challenge for representing many-body quantum wave functions and numerically solving ground state Hamiltonian of quantum many-body systems. However, we can construct an ansatz for $T_{a_1,a_2,...,a_m}$ using tensor networks, which characterize the entanglement property of the system while reducing required computational resources.

Tensor networks are also a useful way to visualize complex operations. Tensors are represented by circles, and indices of those tensors are represented by lines coming out of the circles. For instance a matrix would be represented by a circle with two lines coming out of it, and a rank-3 tensor would be a circle with three lines. To represent "contraction" of tensors, or the summation over certain indices, we can connect lines

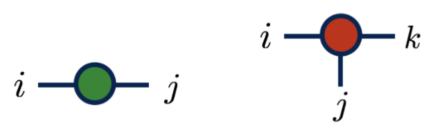
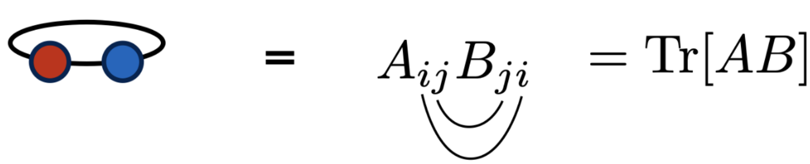


Figure 3: A matrix (green) and a rank-3 tensor (red). All images in this section are thanks to `tensornetwork.org` (insert bib reference)

between different circles. For instance, a scalar product of vectors would be represented with two circles and a line between them. A trace of a matrix product is is two circles connected in a loop:



this formalism can be extended to much more complicated operations, like these. Usefully, tensor networks can also represent quantum states.

4.3 MERA

Different geometries of tensor networks correspond to different entanglement properties of quantum system. Multi-scale entanglement renormalization ansatz (MERA) is a tensor network introduced by Guifre Vidal. [JE21] It is used to represent many-body wavefunctions with long-range entanglement for ground states of gapless Hamiltonians decribed by 1+1 dimensional CFT. In addition, its geometry matches that of anti-de Sitter space. Thus, MERA can be viewed as a discrete realization of the AdS/CFT correspondence.

4.3.1 Correlation Length

Due to constraints imposed by the symmetry under conformal transformation, two point correlation functions for CFT that describes gapless Hamiltonian

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle \propto |x_1 - x_2|^{-q} \ [EV11]. \tag{3}$$

In tensor networks, the correlation function $\langle \phi_1(x_1)\phi_2(x_2)\rangle \propto e^{-\alpha D(x_1,x_2)}$, where $D(x_1, x_2)$ is the distance between x_1 and x_1 in the tensor network. In MERA, $D(x_1, x_2)_{MERA}$ is the length of geodesic connecting which is proportional to $log(x_1 - x_2)$. Thus,

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle_{MERA} \propto e^{-\alpha D_{MERA}(x_1,x_2)} \propto e^{-\alpha log(x_1-x_2)} \propto |x_1 - x_2|^{-q} \ [EV11], \tag{4}$$

which matches the form in (3).

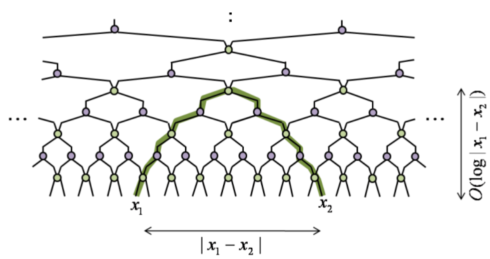


Figure 4: Scaling of $D(x_1, x_2)$ in MERA [EV11]

4.3.2 Entanglement Entropy

Entanglement entropy is often used to describe pure state entanglement of many-body systems. For example, the entanglement of a subsystem A with the rest of the system can be described by

$$S_A = -\mathrm{Tr}(\rho_A \log \rho_A) \ [JE21],$$

where $\rho_A = \mathrm{Tr}_{A^c}(\rho)$ is the reduced density matrix. In 1+1 conformal field theory, the entanglement entropy is

$$S_A = \frac{c}{3} log \frac{l}{a} \ [JE21],$$

where c is the central charge, a is lattice regulator, and l is the length of subsystem A. In tensor networks, S_A scales as the number of bonds that needed to be cut in order to separate subsystem A from the rest of the system. Thus, if A is a boundary region with l indices in MERA, about $log(l)$ bonds need to be cut to separate A from the rest of the system, so $S_A \propto log(l)$, which matches the CFT entanglement entropy scaling. This is reminiscent of the Ryu-Takayanagi formula as shown in (1), with $Area|\gamma_A| \propto log(l)$. Therefore, MERA manifests the properties of the AdS/CFT correspondence.

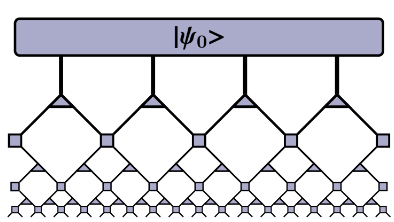


Figure 5: MERA for ground state of Hamiltonian in 1D lattice forms a 2D holographic geometry [JE21]

4.4 Greedy Algorithm

The greedy algorithm described in [Pas+15] is a procedure for determining which areas of the bulk can be mapped to the boundary. In fact, when using this algorithm, the tensor created by contracting over all the legs in the region “found” by the algorithm is guaranteed to be an isometry that maps those incoming legs to the boundary. The algorithm is quite simple. Begin with your tensor P_α. At the beginning P_α is just the bulk degrees of freedom along a specified boundary region A. For any perfect tensor that has at least half of its indices contracted with P_α, add it to P_α and repeat until there are no longer tensors that share enough contracted legs with P_α to qualify. The border of P_α should correspond to the minimal geodesic that connects the ends of A through the bulk. (Note: The algorithm is not perfect, there are cases in which the algorithm fails to find the minimal bulk geodesic. Examples and discussion in [Pas+15] If the algorithm succeeds in finding the minimal geodesic, then the region in ”finds” in the bulk corresponds to the causal wedge of the boundary region A, which means that we can construct operators in that region on A.

4.5 Putting it All Together

The connection between AdS/CFT correspondence and quantum error correcting codes is motivated in [Pas+15] by the following theorems. The numbers inside the parentheses correspond to their numberings in [Pas+15], where their corresponding proofs are also discussed. Additional theorems not discussed in this section will be included in the appendix.

Theorem 1. *(Theorem 1) The pentagon tiling tensor network is an isometric tensor from the bulk to the boundary. We call it the holographic pentagon code.*

Theorem 2. *(Theorem 2) Suppose that we have a holographic state associated to a simply-connected planar tensor network of perfect tensors, whose graph has ”non-positive curvature.” Then for any connected region A on the boundary, we have $SA = |\gamma A|$; in other words, the lattice Ryu-Takyanagi formula holds.*

Theorem 3. *(Theorem 5) Suppose A is a connected boundary region. Then any bulk local operator in the causal wedge $\mathcal{C}[A]$ can be reconstructed as a boundary operator supported on A.*

Theorem 4. *(Theorem 6) Consider a holographic code defined by a contracted network of perfect stabilizer tensors, and suppose that the greedy algorithm starting at the boundary reaches the entire network. Then the code is a stabilizer code.*

From these theorems, we can see a clear path to developing a quantum error correcting code. From Theorem 1 (linked by its numbering in this paper, not in [Pas+15]), we know that the tensor network of pentagons, which takes dangling indices in the bulk as input, is an isometric map to the uncontracted indices on the boundary. From Theorem 2, we know that the Ryu-Takyanagi theorem holds for particular kinds of tensor networks. This specifically connects an explicit property of AdS/CFT to an analagous property in tensor networks of holographic states. The holographic pentagon code. is simply connected and has negative curvature because it is a finite tiling of the uniform tessellation of hyperbolic space, and it is composed of perfect tensors. Because of the specific properties of the tensor networks in question, and because the Ryu-Takyanagi formula holds, we can carry the consequences of Theorem 3 over to these tensor network states. Once we can reconstruct bulk operators on uncontracted boundary legs of a tensor network, we have an error correcting code where the bulk indices are the logical qubits and the boundary indices are the physical qubits. The consequence of Theorem 4 is that we can obtain a specific kind of code called a stabilizer code by building our network out of a specific kind of tensor.

5 The HaPPY code and the [[5,1,3]] Code

5.1 [[5,1,3]] Code

Consider a perfect tensor with 6 indices. By definition, it defines isometries from any set of 0,1,2,3 indices to the complementary set of indices. In fact, these isometries can be considered as encoding maps for [[6,0]],[[5,1]], [[4,2]], [[3,3]] QECC, respectively.

For example, the isometry from 1 index to the complementary 5 indices corresponds to a [[5,1,3]] code that encodes 1 logical qubit into 5 physical qubits. The stabilizer of this [[5,1,3]] code is $S = \langle S_1, S_2, S_3, S_4\rangle$, where

$$
\begin{aligned}
S_1 &= X \otimes Z \otimes Z \otimes X \otimes I \\
S_2 &= I \otimes X \otimes Z \otimes Z \otimes X \\
S_3 &= X \otimes I \otimes X \otimes Z \otimes Z \\
S_4 &= Z \otimes X \otimes I \otimes X \otimes Z
\end{aligned}
$$

The code subspace is the simultaneous eigenspace of these 4 stabilizer generators with eigenvalue +1, i.e, $\mathcal{H}_s = \{|\psi\rangle : S_j|\psi\rangle = |\psi\rangle, j = 1, 2, 3, 4\}$.

5.2 Holographic Code

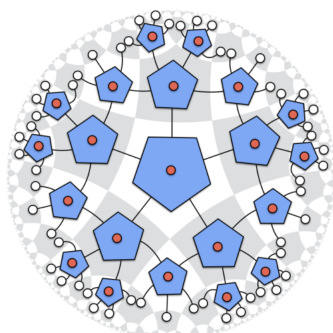


Figure 6: The Holographic Pentagon Code. Image thanks to [Pas+15]

The holographic pentagon code is a tiling of $1 \mapsto 5$ perfect tensors that is tiled over the negatively curved space. It is essential that the surface be negatively curved so 4 pentagons can meet at a vertex, this is impossible in flat space. This specific geometry allows the greedy algorithm to accurately find the causal wedge, and it helps "protect," in a sense, qubits further into the bulk because each tensor will have at most two legs contracted with the previous layer. Interpreting this diagram, the red dots are logical qubits, the blue pentagons are the perfect tensors, and the white dots on the outside are the physical qubits. As explained by John Preskill, one of the creators of the code,"the red dots in the bulk [are] a code space which is embedded in the Hilbert space of those [white] boundary qubits." [Pre16a] (39:00) The code does not have to be of the exact form shown in the figure. In fact, just one of these tensors constitutes a [[5,1,3]] code. Their successive layering can encode more logical qubits in more physical qubits on the boundary through the process of concatenation. As the number of layers grows large, the rate of the code ($\frac{\text{Number of bulk indices}}{\text{Number of boundary indices}}$) approaches $\dfrac{1}{\sqrt{5}}$. ([Pas+15] (Appendix c)

5.3 Proof of Protection

How effective is the holographic code at protecting against erasure errors? With modifications, it can be extremely effective, but the use of only pentagons across the entire bulk creates a problem. Because each tensor in the layer just inside the boundary can have three or four dangling indices on the boundary, and because the greedy algorithm that determines the causal wedge, and in turn the region of the bulk where operators can be reconstructed, crosses a cut only when a tensor has at least half of its indices contracted by P_α, it is possible to erase specific qubits on the boundary to ensure that the causal wedge of the unerased boundary regions never reaches the center. So in general, the code may be useful, but it is impossible to argue this analytically because of the existence of a small group of specific errors that can prevent reconstruction of the central bulk operator. [Pas+15] To better prevent the propagation of errors deep into the bulk, we can substitute some of the bulk tensors with dangling indices with hexagons, which have every index contracted. For graphs of certain kinds you can analytically determine an "erasure threshold," or a critical number of boundary qubit erasures that prevent reconstruction of some important bulk operator. There is such an argument for the following code, which has only one central qubit, but the same analysis can be applied to other networks. A more in-depth explanation of the technique is available in [Pas+15] Appendix D, but

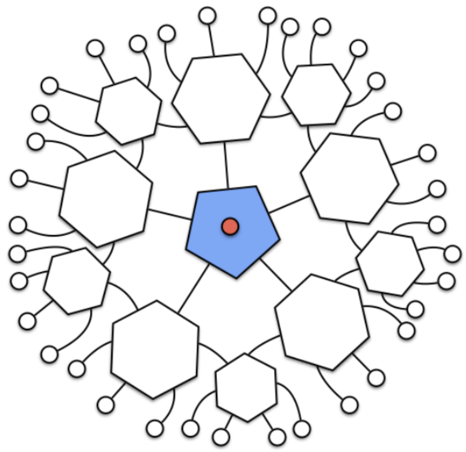


Figure 7: A very special qubit. Image thanks to [Pas+15]

intuitively, it involves recursively bounding a boundary erasure probability. Because each tensor has at least four legs contracted at a layer further out, to "lose" that tensor, you have to "lose" at least two legs of a tensor one layer further out. There are only three ways for this to happen, because we only care about losing neighboring tensors. If the tensors we lose are not next to each other, in this geometry, then the error will be contained, because there are more boundary indices per tensor in the second-to-last layer. Encoding this fact in recursively defined inequalities and solving leads to a critical erasure threshold of $\frac{1}{12}$ for this specific geometry.

6 Open Problems

One problem that is currently being investigated within error correction is whether it is possible to correct other kinds of errors than erasure errors, specifically phase errors, using tensor networks. There is also lots of investigation into tensor networks themselves, and what kind of algorithms and useful results for many-body physics can be obtained from them. Another problem is how to use tensor networks to describe different kinds of many body systems, and how to apply tensor networks when the number of bodies in a system becomes very large. It is also interesting to ask whether these tensor networks might be extended to be able to usefully describe complicated dynamic processes, as right now they cannot describe any time evolution of systems.

The holographic QECC also provides a way to describe black holes. For example, by removing the central tensor in the pentagon code described in section 5, and assigning the original boundary indices of the central tensor to be additional bulk indices, we will be able to construct a QECC with larger code subspace, and we can interpret this code subspace as a description of bulk with black hole in the middle. Moreover, tensor networks could potentially be used to describe configurations of worm-wholes, and exploring the relationship between the length of worm hole with the complexity of tensor networks would be an interesting future direction.

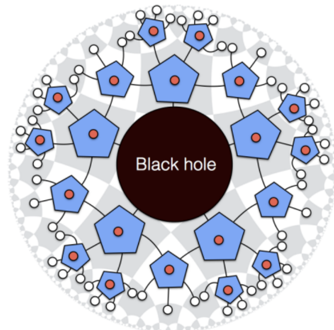


Figure 8: A mysterious black hole. Image thanks to [Pas+15]

7 Appendix

7.1 Maximally Symmetric Space

7.1.1 Homogeneous

When a space M is homogeneous: $\forall$ pairs of points $p, q \in M$, $\exists$ an isometry $\phi \in I(M)$ (the group of all isometries) s.t. $\phi(p) = q$.

7.1.2 Isotropic

When a space is isotropic: $\forall$ points p $\in M$ and for any two tangent vectors $v, w \in T_pM$ s.t. $|v| = |w|, \exists$ an isometry $\phi \in I(M)$ s.t. $\phi(p) = p$ and $\phi_*(v) = w$.

7.1.3 Killing Vectors

Killing vectors corresponds to symmetries of the metric. A metric that has time translational symmetry, for example, has a Killing vector of (1,0,0,0). In general, a vector field K is called a Killing vector if it satisfies $\nabla_\alpha K_\beta + \nabla_\beta K_\alpha = 0$, where $\nabla_\alpha K_\beta = g_{\mu\beta}\nabla_\alpha K^\mu = g_{\mu\beta}(\partial_\alpha K^\mu + \Gamma^\mu_{\alpha\lambda}K^\lambda)$ is the covariant derivative.

7.2 Tensors

7.2.1 Isometric Tensors

A tensor is isometric if it obeys the following property:

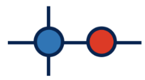
$$\sum T^\dagger_{a'b} T_{ba} = \delta_{a'a}$$

where T is a two index tensor which acts like

$$T : |a\rangle \mapsto \sum_b |b\rangle\, T_{ba}$$

and $|a\rangle, |b\rangle$ are states in the Hilbert spaces $\mathcal{H}_A, \mathcal{H}_B$, respectively.

7.2.2 Tensor Network Operations



$$= \sum_k T_{ijkl} V_{km}$$

$$= \sum_{\alpha_1,\alpha_2,\alpha_3} A^{s_1}_{\alpha_1} B^{s_2}_{\alpha_1\alpha_2} C^{s_3}_{\alpha_2\alpha_3} D^{s_4}_{\alpha_3}$$

Some more examples of more involved tensor contractions shown using tensor networks. Image thanks to `tensornetwork.org`

7.3 Additional Theorems from [Pas+15]

Theorem 5. *(Theorem 3) For a holographic state or code, if* A *is a (not necessarily connected) boundary region and* A^C *is its complement, then the entropy of* A *satisfies* $SA \geq |\gamma^*_A \cap \gamma^*_{A^C}|$*, where* γ^*_A *is the greedy geodesic obtained by applying the greedy algorithm to* A *and* $\gamma^*_{A^C}$ *is the greedy geodesic obtained by applying the greedy algorithm to* A^C*.*

Theorem 6. *(Theorem 4) Suppose the* $2n$ *indices of a perfect tensor state are partitioned into four disjoint nonempty sets* A, B, C, D *such that* $0 < |A|, |B|, |C|, |D| < n$*. Then the tripartite information* I_3 *is strictly negative:* $I_3(A, B, C) < 0$*.*

7.4 Fatal Errors

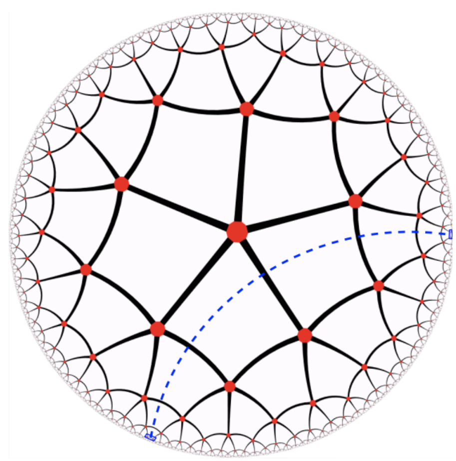


Figure 9: Fatal Boundary Errors. Image thanks to [Pas+15]

The above image shows four boundary erasures that would prevent the causal wedge of the area between the erasures from ever reaching the center of the bulk. If these specific, adjacent errors happened in 3 or more places, it could become impossible to reconstruct the central logical qubit on any subset of the boundary qubits.

7.5 Geodesics

Geodesics are curves in curved space analogous of straight lines in flat space. They are curves with smallest distance between two points, i.e, their tangent vectors $\frac{dx^\alpha}{d\lambda}$ satisfy $\frac{D}{D\lambda}\frac{dx^\alpha}{d\lambda} = 0$, where $\frac{D}{D\lambda} = \frac{dx^\mu}{d\lambda}\nabla_\mu$ is the directional derivative along the curve.

7.6 Abelian Subgroup

An abelian subgroup is a subset that is closed under a commutative group operation, contains the identity element, and every element in it has an inverse.

7.7 Gapless Hamiltonian

A Hamiltonian is called gapless if there is no infinite separation between the energy of the ground state and first excited state, i.e, the energy spectrum is "continuous".

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