COMS 6998 - Introduction to Quantum Computing	Spring 2022

Lecture 4 - POVMs and Pure State Tomography

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1 Generalized Measurements

We've previously discussed standard basis measurements. Recall that if you have a d-dimensional state $|\psi\rangle$ and you measure in the standard basis, then the probability of an outcome $a \in [d]$ is

$$\Pr_{|\psi\rangle}[a] = |\langle a|\psi\rangle|^2 .$$

However, there are more general kinds of measurements that allow us to measure in a basis other than the standard basis. We call these measurements *projective measurements*.

A set of matrices $M = \{M_1, \ldots, M_k\}$ is a *k-outcome projective measurement* if each M_a is a projector and $M_1 + \cdots + M_k = I$, the identity matrix. The probability of obtaining outcome $a \in [k]$ when measuring a state $|\psi\rangle$ with projective measurement M is defined to be $\langle\psi|M_a|\psi\rangle = \mathrm{Tr}(M_a|\psi\rangle\langle\psi|)$.

For example, if we want to measure in the orthogonal basis $\{|b_j\rangle\}$, an orthogonal basis for $\mathbb{C}^d$, then the corresponding projective measurement would be $M = \{M_a\}_{a\in[d]}$ where $M_a = |b_a\rangle\langle b_a|$. A projective measurement of this form can be implemented using a unitary transformation plus a standard basis measurement in the following way: first, apply a unitary U to $|\psi\rangle$ where U is a unitary that maps $|b_a\rangle$ to the standard basis vector $|a\rangle$. Then measure $U|\psi\rangle$ in the standard basis. We have that the probability of obtaining outcome a is equal to

$$|\langle a|U|\psi\rangle|^2 = |\langle b_a|\psi\rangle|^2$$

which is the same as if you directly measured $|\psi\rangle$ using the projective measurements $\{|b_a\rangle\langle b_a|\}_a$.

1.1 Positive Operative Value Measure (POVM)

The is an even more general type of measurement we can perform on a quantum state $|\psi\rangle \in \mathbb{C}^d$. Suppose we do the following:

- Append a qubit to form the state $|\psi\rangle \otimes |0\rangle$.
- Measure the enlarged system $|\psi\rangle \otimes |0\rangle$ using a projective measurement $M = \{M_1, \ldots, M_k\}$ that acts on the larger space $\mathbb{C}^d \otimes \mathbb{C}^2$.

The probability of obtaining outcome $a \in [k]$ is, according to the foregoing discussion:

$$\Pr[a] = (\langle\psi| \otimes \langle 0|)\, M_a\, (|\psi\rangle \otimes |0\rangle) .$$

We can write with this probability in terms of $|\psi\rangle$ and some other matrix Q_a. We label the systems in $\mathbb{C}^d$ and $\mathbb{C}^2$ as A and B, respectively, and "bring out" the $|\psi\rangle$. We can rewrite our final outcome probability equation as

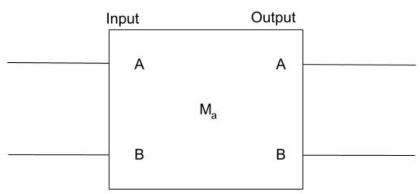
$$\begin{aligned}\Pr[a] &= (\langle\psi|_A \otimes \langle 0|_B)\, M_a\, (|\psi\rangle_A \otimes |0\rangle_B) \\ &= \langle\psi|_A\, (I_A \otimes \langle 0|_B)\, M_a\, (I_A \otimes |0\rangle_B)\, |\psi\rangle_A \\ &= \langle\psi|_A\, Q_a\, |\psi\rangle_A\end{aligned}$$

where we've defined Q_a to be the matrix

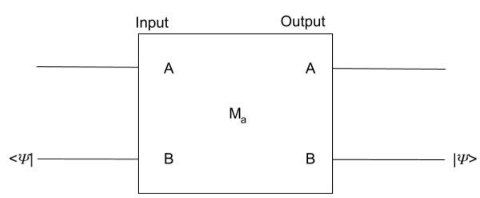
$$Q_a = (I_A \otimes \langle 0|_B) M_a (I_A \otimes |0\rangle_B)|\psi\rangle_A \ .$$

Note that this is a matrix that acts on the A system only. In other words, Q_a is a $d \times d$ matrix.

One way to understand this is that we can visualize M_a as a box with two "wires" that transforms an input in the $A \otimes B$ space into an output in that same space. Each of these two wires acts on one of these two spaces.



The operator Q_a can be thought of us "capping" the B wire (by multiplying it by the qubit vector $|0\rangle$) of M_a so that the resulting operator takes inputs from the A space and produces outputs in the A space.



Another way of thinking about Q_a is that, in the appropriate basis, it is the upper left block of M_a:

$$M_a = \begin{pmatrix} Q_a & \cdots \\ \cdots & \cdots \end{pmatrix} \ .$$

The set of matrices $Q = \{Q_a\}$ formed in this way from the projective matrices $\{M_a\}$ is called a *Positive Operative Value Measure (POVM)*. More formally, a *k-outcome POVM* is a set $Q = \{Q_1, \ldots, Q_k\}$ of matrices such that Q_a is PSD for all $a \in [k]$ and $\sum_a Q_a = I$. The probability of obtaining outcome a when measuring $|\psi\rangle$ is equal to $\langle\psi|Q_a|\psi\rangle$.

2 Continuous POVMs

So far measurements only have finitely many outcomes. What if we want to talk about measurements that return a value from an infinite, or even continuous, set? For example, one can imagine performing a measurement on a particle to determine its location. We would expect a real number out.

Let Ω be an outcome space (which is potentially infinite) with a *measure* $\mathrm{d}x$, which intuitively is a way of assigning numbers to various region within the space (such as length, area, volume, etc. depending on dimension). Then a *continuous POVM* over Ω is a collection of matrices $Q = \{Q_x\}_{x\in\Omega}$ such that each Q_x is PSD and

$$\int Q_x \, \mathrm{d}x = I \ .$$

We can compare this to its discrete POVM counterpart, $\sum_x Q_x = I$.

Suppose we measure a state $|\psi\rangle$ with a continuous POVM Q. If Ω is an infinite space, such as the real line $\mathbb{R}$, then intuitively we expect the probability of obtaining any specific outcome $x \in \Omega$ to be 0 (just like how the probability of sampling any specific real number from the Gaussian distribution is 0); instead we measure the probability of obtaining an outcome in a (measurable) region $S \subseteq \Omega$:

$$\Pr_{|\psi\rangle}[x \in S] = \int_S \langle\psi| \, Q_x \, |\psi\rangle \, \mathrm{d}x \ .$$

Furthermore, suppose we have a function $f : \Omega \to \mathbb{R}$ and we want to determine the *average* value of f if we measure a state $|\psi\rangle$, obtain value $x \in \Omega$, and evaluate $f(x)$:

$$\mathbb{E}_{|\psi\rangle}[f(x)] = \int_\Omega \langle\psi| \, Q_x \, |\psi\rangle \, f(x) \, \mathrm{d}x \ .$$

3 Pure State Tomography

Recall that we've previously discussed a simple tomography algorithm for mixed states; it has sample complexity $\tilde{O}(d^6)$ where d is the dimension. We will now discuss an algorithm for pure state tomography that is more efficient: it only requires $O(d)$ copies of the input state. This comes close to the lower bound of $\Omega\left(\frac{d}{\log d}\right)$ that we proved.

Suppose we are performing tomography on d-dimensional pure states. Define the outcome space Ω to be $S(\mathbb{C}^d)$, the set of unit vectors in $\mathbb{C}^d$. This is naturally endowed with the Haar measure, which we denote by $\mathrm{d}\theta$. Define the continuous POVM $Q = \{Q_{|\theta\rangle}\}_{|\theta\rangle\in\Omega}$ as follows:

$$Q_{|\theta\rangle} = |\theta\rangle\langle\theta|^{\otimes k} \binom{k+d-1}{k} \ .$$

Note that there is a matrix $Q_{|\theta\rangle}$ for every $|\theta\rangle \in \mathbb{C}^d$, and the matrix acts on the space $(\mathbb{C}^d)^{\otimes k}$. The matrix $Q_{|\theta\rangle}$ is clearly PSD, and furthermore

$$\int Q_{|\theta\rangle} \, \mathrm{d}\theta = \binom{k+d-1}{k} \int |\theta\rangle\langle\theta|^{\otimes k} \, \mathrm{d}\theta = \binom{k+d-1}{k} \frac{P^{\mathrm{sym}}_{d,k}}{\mathrm{Tr}(P^{\mathrm{sym}}_{d,k})} = P^{\mathrm{sym}}_{d,k}$$

where we used our formula for integrating $|\theta\rangle\langle\theta|^{\otimes k}$ over the Haar measure. One might be worried that this is not a valid continuous POVM because the integral is not the identity matrix. However, for all intents and purposes it is, because we are only going to measure states of the form $|\psi\rangle^{\otimes k}$, which is a member of the symmetric subspace $\mathrm{Sym}(d,k)$, for which the projection $P^{\mathrm{sym}}_{d,k}$ *is* effectively the identity matrix. Thus we can treat Q as a valid continuous POVM.

The algorithm. Given $|\psi\rangle^{\otimes k}$ for some $|\psi\rangle \in \mathbb{C}^d$, we perform the continuous POVM Q on it to obtain an outcome $|\theta\rangle \in \mathbb{C}^d$. Ideally, this outcome should be equal to $|\psi\rangle$ but it won't be exactly. How close of an estimate is it? We can measure this by considering the *squared overlap* between $|\psi\rangle$ and $|\theta\rangle$. Define the function $f(|\theta\rangle) = |\langle\theta|\psi\rangle|^2$ where $|\psi\rangle$ is the unknown state. We want to know what $f(|\theta\rangle)$ is on average. According to our formula:

$$\begin{aligned}
\mathbb{E}\left[f(x)\right] &= \int \langle\psi|^{\otimes k}\, Q_{|\theta\rangle}\, |\psi\rangle^{\otimes k}\, f(|\theta\rangle)\,\mathrm{d}\theta \\
&= \binom{k+d-1}{k} \int \langle\psi|^{\otimes k} \left(|\theta\rangle\langle\theta|^{\otimes k}\right) |\psi\rangle^{\otimes k} \cdot |\langle\theta|\psi\rangle|^2\,\mathrm{d}\theta && \text{(Definition of } Q_{|\theta\rangle}) \\
&= \binom{k+d-1}{k} \int \langle\psi|^{\otimes k+1} \left(|\theta\rangle\langle\theta|^{\otimes k+1}\right) |\psi\rangle^{\otimes k+1}\,\mathrm{d}\theta \\
&= \binom{k+d-1}{k} \langle\psi|^{\otimes k+1} \left(\int |\theta\rangle\langle\theta|^{\otimes k+1}\,\mathrm{d}\theta\right) |\psi\rangle^{\otimes k+1} \\
&= \binom{k+d-1}{k} \langle\psi|^{\otimes k+1} \cdot \frac{P^{\mathrm{sym}}_{d,k+1}}{\mathrm{Tr}(P^{\mathrm{sym}}_{d,k+1})} |\psi\rangle^{\otimes k+1} && \text{(Formula for integral)} \\
&= \binom{k+d-1}{k} \cdot \binom{k+d}{k+1}^{-1} && \text{(Dimension of } P^{\mathrm{sym}}_{d,k+1}) \\
&= \frac{(k+d-1)!}{k!(d-1)!} \cdot \frac{(k+1)!(d-1)!}{(k+d)!} \\
&= \frac{k+1}{k+d}\,.
\end{aligned}$$

Suppose we set $k = d/\varepsilon$. Then this quantity is $1 - O(\varepsilon)$, which means that on average the output of the tomography algorithm will have high overlap with the unknown input state $|\psi\rangle$.