Quantum Non-Local Games and Representation Theory

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Abstract

Quantum non-local games have been an active area of research in the past few years. There have been applications of non-local games to verification of quantum systems, and the study of quantum interactive proofs. They are also interesting from the perspective of mathematical physics and pure mathematics, as certain concepts from functional analysis and representation theory have been used in their study. In this project paper, we will overview results that relate quantum non-local games to the approximate representation theory of their solution groups.

1 Introduction

Non-local games have been extensively studied in quantum information theory. We have considered numerous applications of non-local games in this class. For instance, the CHSH game has been used to demonstrate that there are indeed differences between classical and quantum mechanics in physics [CHSH69]. In computer science, quantum non-local games can be used as part of a protocol that enables a classical polynomial time machine to verify the results of a quantum computation, assuming we have two (possibly untrusted) quantum devices that can possibly share entanglement with each other [Gri17].

In a breakthrough result proven earlier this year, it was shown that there was no algorithm to approximate the maximum winning probability of a non-local game assuming that the players use a quantum strategy. This has the consequence of showing that $\mathsf{MIP^*} = \mathsf{RE}$ [JNV$^+$20], that the problems verifiable by multi-prover quantum interactive proofs can be characterized exactly by the class of recursively enumerable problems. In other words, a classical polynomial-time verifer can verify whether or not a Turing machine halts, an otherwise undecidable problem, assuming interaction with two entangled quantum provers! More strikingly, the complexity-theoretic result that $\mathsf{MIP^*} = \mathsf{RE}$ resolves two long-standing open problems in mathematics. In particular, it implies a negative result for Tsirelson's problem in mathematical physics that compares two models of quantum mechanics, which also gives a negative result for Connes' Embedding Conjecture in the theory of von Neumann algebras.

In this paper, our focus is to study tools from group theory and representation theory that have applications in the theory of non-local games and the study of Connes' Embedding Conjecture. The organisation of the paper is as follows: we introduce the basics in Section 2, by defining a simple class of non-local games called linear system games, what is meant by quantum strategies for such games, and their *solution groups*. Section 3 forms the technical core of the paper, in which we investigate the relationship between the approximate representation theory of solution groups and perfect quantum strategies. Finally in Section 4, we discuss additional notions such as amenable, sofic and hyperlinear groups, their connections to *rigidity* of non-local games and end with some interesting open problems.

2 Linear System Games

We will focus on a class of non-local games called linear system games.

Definition 1. Let $Ax = b$ be a system of equations where $A \in \mathbb{Z}_2^{m\times n}$ is an $m \times n$ binary matrix and $b \in \mathbb{Z}_2^m$ is a binary vector. The linear system game associated to $Ax = b$ is the non-local game defined as follows:

- **Inputs:** Alice receives as input an equation $i \in \{1, \ldots, m\}$ specified by row i in A, and Bob receives a variable $j \in \{1, \ldots, n\}$.

- **Outputs:** Alice and Bob must respond with assignments to the variables they receive. They win the non-local game if Bob's variable does not appear in Alice's equation or else, Alice's assignment satisfies equation i, and Bob and Alice's assignment are consistent on the variable x_j.

By convention we will use $\mathcal{I}_A, \mathcal{I}_B$ to denote the input sets of Alice and Bob respectively and $\mathcal{O}_A, \mathcal{O}_B$ to denote the output sets of Alice and Bob respectively. We also define $V_i := \{j \mid A_{ij} \neq 0\}$ i.e., the set of variables appearing in equation i.

An example of a linear system game is the magic square game, based on the following system of equations with 6 equations and 9 variables.

$$\begin{aligned} x_1 + x_2 + x_3 = 0 \quad x_4 + x_5 + x_6 = 0 \quad x_7 + x_8 + x_9 = 0 \\ x_1 + x_4 + x_7 = 1 \quad x_2 + x_5 + x_8 = 1 \quad x_3 + x_6 + x_9 = 1 \end{aligned} \tag{1}$$

From adding together the equations shown, it is immediate that there is no solution $x_i \in \{0, 1\}$ that can satisfy the linear system. This means that the probability that Alice and Bob can win the game using a classical strategy assuming that equations and variables are chosen uniformly at random is strictly less than one. However, it turns out that Alice and Bob can win the game perfectly should they use a quantum strategy and share quantum entanglement between them! This is perhaps a suprising result and has been called "quantum pseudo-telepathy" in the physics literature. Such a strategy is described in our course notes [Yue20].

We now define more precisely what it means for Alice and Bob to have a quantum strategy for the linear system game.

Definition 2 (Classes of Quantum Correlations)**.** The following are different classes of quantum strategies we will consider.

- A strategy is a **tensor product quantum strategy** if there are Hilbert spaces $\mathcal{H}_A, \mathcal{H}_B$, a state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, projective measurements for Alice $\{P_x^a\}$ for every $x \in \mathcal{I}_A$ and projective measurements for Bob $\{Q_y^b\}$ for every $y \in \mathcal{I}_B$ such that the probability that Alice and Bob output $(a, b) \in \mathcal{O}_A \times \mathcal{O}_B$ given input $(x, y) \in \mathcal{I}_A \times \mathcal{I}_B$ satisfies $p(a, b|x, y) = \langle\psi| P_x^a \otimes Q_x^b |\psi\rangle$. That is, Alice and Bob obtain their output by measuring the quantum state using their respective measurements. In particular, we say that the strategy is a **finite-dimensional** strategy if $\mathcal{H}_A$ and $\mathcal{H}_B$ are finite dimensional. We write $\mathcal{C}_q$ for the set of correlations (i.e. the tensors p) that can be obtained by finite-dimensional strategies and $\mathcal{C}_{qs}$ for the set of correlations that can be obtained by tensor-product strategies.
- We will write $\mathcal{C}_{qa}$ for the **closure** of $\mathcal{C}_q$, the set of finite dimensional correlations.
- A strategy is a **commuting operator strategy** if there is a Hilbert space $\mathcal{H}$, a state $|\psi\rangle \in \mathcal{H}$, projective measurements for Alice $\{P_x^a\}$ for every $x \in \mathcal{I}_A$ and projective measurements for Bob $\{Q_y^b\}$ for every $y \in \mathcal{I}_B$ such that $P_x^a Q_y^b = Q_y^b P_x^a$ such that the probability that Alice and Bob output $(a, b) \in \mathcal{O}_A \times \mathcal{O}_B$ given input $(x, y) \in \mathcal{I}_A \times \mathcal{I}_B$ satisfies $p(a, b|x, y) = \langle\psi| P_x^a Q_x^b |\psi\rangle$. We will write $\mathcal{C}_{qc}$ for set of correlations that can be realized by the set of commuting operator strategies.
- The $\mathcal{C}_t$ **value** $\omega_t^*(G)$ of a game G for $t \in \{q, qs, qa, qc\}$ is the maximum winning probability of the game G when Alice and Bob use correlations in $\mathcal{C}_t$ to obtain their outputs. In equations, if in a game G, questions (x, y) are sampled according to distribution $\Pi(x, y)$, and $D(a, b|x, y) = 1$ if the tuple (x, y, a, b) wins the game (and is zero otherwise), then

$$\omega_t^*(G) := \sup_{p \in \mathcal{C}_t} \sum_{(x,y) \in \mathcal{I}_A \times \mathcal{I}_B} \Pi(x, y) \sum_{(a,b) \in \mathcal{O}_A \times \mathcal{O}_B} D(a, b|x, y) p(a, b|x, y).$$

  Moreover, we say that a game G has a perfect $\mathcal{C}_t$-strategy if $\omega_t^*(G) = 1$.

We have the inclusions that $\mathcal{C}_q \subseteq \mathcal{C}_{qs} \subseteq \mathcal{C}_{qa} \subseteq \mathcal{C}_{qc}$. The proofs of these inclusions can be found in [SW08] and whether or not these inclusions are strict is known as Tsirelson's problem. *Separating these sets of correlations would mean that there can be fundamental differences between different models of quantum mechanics used in physics.*

A key observation is that the strictness of these inclusions can be witnessed by non-local games. That is, if we can construct a non-local game G where $\omega_t^*(G) < \omega_{t'}^*(G)$ for two different sets of quantum correlations then $\mathcal{C}_t \neq \mathcal{C}_{t'}$. In other words, a non-local game provides a linear functional that will show that the two sets are different should its optimal value be different between the two sets. We now describe how the *representation theory of groups* can be used to construct linear system games that witness these separations, in particular the separation that $\mathcal{C}_{qs} \subsetneq \mathcal{C}_{qa}$ proven by Slofstra [Slo19].

2.1 The Solution Group of a Linear System Game

For linear system games, since the output is binary, it is easier to think of a strategy in terms of observables rather than projective measurements. Note that we will also switch from writing the system of equations additively, where variables take values $\{0, 1\}$, to writing the system of equations multiplicatively, where variables take values $\{-1, 1\}$. For example, we write the equation $x_1 + x_2 + x_3 = 1$ as the product $x_1 x_2 x_3 = -1$.

Definition 3. A **tensor product** quantum strategy for a linear system game is specified by the following data:

1. A Hilbert space $\mathcal{H}_A$ for Alice and a Hilbert space $\mathcal{H}_B$ and a state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$,
2. A set of self-adjoint operators X_j on $\mathcal{H}_B$ satisfying $X_j^2 = I$ for each variables $1 \leq j \leq n$, one for each variable that Bob receives.
3. A set of self-adjoint operators Y_{ij} satisfying $Y_{ij}^2 = I$ on $\mathcal{H}_A$ for each equation i and variable j. We furthermore assume that Alice's measurements are locally compatible so that $Y_{ij} Y_{il} = Y_{il} Y_{ij}$ for all variables j, l appearing in equation i and additionally $\prod_{j \in V_i} Y_{ij} = (-I)^{b_i}$ for every V_i so that Alice's answers will satisfy the constraints given and are independent of the order she measures her observables.

The above definition can be extended to present a commuting operator quantum strategy in terms of observables. The presentation in terms of observables then leads to a presentation in terms of measurement operators previously described by computing the spectral decomposition of the operators in the strategy.

We now encode the requirements that a quantum strategy must satisfy in terms of an algebraic object called the solution group of the linear system game. This notion was introduced in [CLS17]. As an intermediate step, we define the notion of an operator solution of $Ax = b$. We will always assume that the system has m equations and n variables.

Definition 4. An **operator solution** of $Ax = b$ is a set of bounded self-adjoint operators $X_1, \ldots, X_n$ on a Hilbert space $\mathcal{H}$ (i.e. an assignment of an operator to each variable) where

1. $X_j^2 = I$ for all $1 \leq j \leq n$
2. The operators are locally compatible. That is, if x_i, x_j are variables appearing the same equation, then $X_i X_j = X_j X_i$
3. The operators satisfy the equation $\prod_{j=1}^{n} X_j^{A_{ij}} = (-I)^{b_i}$ for each equation $1 \leq i \leq m$.

Now, the solution group is a group defined with $n+1$ generators (i.e. the variables and extra generator J taking the place of -1), intended to capture all relations among the variables that an operator solution must satisfy. We use the notation $[a, b] = aba^{-1}b^{-1}$ for the commutator of elements in a group.

Definition 5. The **solution group** $\Gamma(A, b)$ of a linear system $Ax = b$ is the group generated by $n + 1$ elements $x_1, \ldots, x_n, J$ satisfying the following relations:

1. Each generator is an involution: $x_j^2 = 1$ for $1 \leq j \leq n$, and $J^2 = 1$.
2. J commutes with every generator: $[x_j, J] = 1$ for all $1 \leq j \leq n$.
3. If x_i, x_j appear in the same equation, then $[x_i, x_j] = 1$.
4. $\prod_j x_j^{A_{ij}} = J^{b_i}$ for each equation $1 \leq i \leq m$.

The solution group is an abstract group with certain generators and relations. However, once we have a representation of the group, we can then extract a quantum strategy for Alice and Bob. For instance, the following theorems relate perfect strategies for the LCS game in $\mathcal{C}_q, \mathcal{C}_{qs}$ and $\mathcal{C}_{qc}$ to the solution group.

Theorem 1 ([CM13, CLS17])**.** *Let G be the linear system game associated with a set of equations $Ax = b$. The following are equivalent:*

- *Alice and Bob have a perfect finite-dimensional quantum strategy: $\omega_q^*(G) = 1$.*
- *Alice and Bob have a perfect tensor product strategy: $\omega_{qs}^*(G) = 1$.*
- *There is a finite-dimensional representation ρ of the solution group $\Gamma(A, b)$ where $\rho(J) \neq I$.*
- *There is a finite-dimensional operator solution to $Ax = b$.*

A similar theorem can also be proven for commuting operator strategies.

Theorem 2 ([CLS17])**.** *Let G be the linear system game associated with a set of equations $Ax = b$. The following are equivalent:*

- *Alice and Bob have a perfect commuting operator strategy: $\omega_{qc}^*(G) = 1$.*
- *$J \neq 1$ in the solution group $\Gamma(A, b)$.*
- *There is an operator solution to $Ax = b$.*

A characterization theorem for when $\omega_{qa}^*(G) = 1$ is more involved and involves looking at approximate representations of the solution group, which will be introduced in the next section.

3 Separating Sets of Quantum Correlations

We devote this section to providing an overview of the result that $\mathcal{C}_{qs} \subsetneq \mathcal{C}_{qa}$ proven by Slofstra [Slo19] i.e., there are non-local games (with finite classical input and output sets) that cannot be played optimally using any fixed dimension. First, let us set up some group theory notation.

Given a set S, let $\mathcal{F}(S)$ denote the free group generated by S. If R is a subset of $\mathcal{F}(S)$, then the quotient of $\mathcal{F}(S)$ by the normal subgroup generated by R is denoted by $\langle S : R \rangle$. A group G is said to be *finitely presentable* if $G = \langle S : R \rangle$ for some finite sets S and R. A *finitely presented group* is a tuple (G, S, R), where $G = \langle S : R \rangle$. In other words, a finitely presented group is a finitely presentable group along with a choice of finite presentation.

3.1 Approximate Representations

Let $\|\cdot\|$ be the normalized Hilbert-Schmidt norm, i.e. if T is an endomorphism of a finite-dimensional Hilbert space $\mathcal{H}$, then $\|T\| = \sqrt{\mathrm{tr}(T^*T)}/\sqrt{\dim \mathcal{H}}$. The notion of approximate representations is fairly intuitive: while a representation of a finitely presented group is a homomorphism from the corresponding free group to the unitary group of a Hilbert space that also preserves the relations specified by the presentation, an approximate representation is one where the relations are only *approximately* preserved.

Definition 6. Let $G = \langle S : R \rangle$ be a finitely presented group. A *finite-dimensional ε-approximate representation* (or *ε-representation* for short) is a homomorphism $\phi : \mathcal{F}(S) \to \mathcal{U}(\mathcal{H})$ from $\mathcal{F}(S)$ to the unitary group $\mathcal{U}(\mathcal{H})$ of some finite-dimensional Hilbert space $\mathcal{H}$, such that for all $r \in R$,

$$\|\phi(r) - I\| \leq \varepsilon.$$

We need the next definition in order to specify a partial $\mathcal{C}_{qa}$ analogue to Theorems 1 and 2.

Definition 7. Let G be a finitely presentable group. An element $g \in G$ is *non-trivial in (finite-dimensional) approximate representations* if there is a finite presentation $G = \langle S : R \rangle$, a representative $w \in \mathcal{F}(S)$ for g, and some constant $\delta > 0$ such that, for all $\varepsilon > 0$, there is an ε-representation ϕ of G with $\|\phi(w) - I\| > \delta$.

Moreover, the choice of presentation $\langle S : R \rangle$ and representative w in Definition 7 is arbitrary. This is not completely obvious from the definition, but we shall skip the proof in the interest of brevity. The following theorem is the main connection between approximate representation theory and perfect $\mathcal{C}_{qa}$-strategies that Slofstra exploits in order to separate $\mathcal{C}_{qa}$ from $\mathcal{C}_{qs}$.

Theorem 3. *Let $\Gamma = \Gamma(A, b)$ be a solution group. If J_Γ is non-trivial in finite-dimensional approximate representations of Γ, then $\omega_{qa}^*(A, b) = 1$.*

Given Theorems 1 and 3, note that in order to prove $\mathcal{C}_{qs} \neq \mathcal{C}_{qa}$, it suffices to construct a solution group Γ such that J_Γ is trivial in finite-dimensional representations, but non-trivial in approximate representations. This construction is carried out in [Slo19] by constructing explicit games (involving 235 variables and 184 equations) whose solution group belongs to a broad class of groups which he calls *linear-plus-conjugacy groups*. This construction itself is not too technically challenging and involves embedding a *sofic* group (see Section 4 for a definition) into a solution group. But given that our focus in this paper is to explore the connection between *representation theory* and quantum strategies, we restrict ourselves here to only proving Theorem 3.

3.2 Deconstructing the Proof of Theorem 3

In this subsection, we provide an earnest attempt to deconstruct the proof of Theorem 3 that is given in [Slo19]. That is, instead of a standard regurgitation of the argument in [Slo19], we attempt to motivate every step along the way, and elucidate why the notion of approximate representations from Defintion 6 really is the right thing to associate with perfect $\mathcal{C}_{qa}$ strategies.

Let us first analyze what perfect strategies in $\mathcal{C}_{qa}$ correspond to. Recall that a tensor product strategy for $Ax = b$ is a tuple $(\{Y_{ij}\}, X_j, |\psi\rangle)$ that satisfies the conditions in Definition 3. A simple calculation shows that if $j \in V_i$, then $\langle\psi| Y_{ij} \otimes X_j |\psi\rangle = 2p_{ij} - 1$, where p_{ij} is the probability with which Alice and Bob win on inputs i and j.

Next, we claim that $\omega^*_{qa}(A, b) = 1$ if and only if, for all $\varepsilon > 0$, there is a finite-dimensional quantum strategy $(\{Y_{ij}\}, X_j, |\psi\rangle)$ such that

$$\langle\psi| Y_{ij} \otimes X_j |\psi\rangle \geq 1 - \varepsilon \text{ for all } 1 \leq i \leq n, j \in V_i.$$

To see this, note that $\mathcal{C}_{qa}$ is defined as the closure of $\mathcal{C}_q$. The linear system game associated to $Ax = b$ has a perfect strategy in $\mathcal{C}_{qa}$ if and only if, for every $\varepsilon > 0$, there is a finite-dimensional quantum strategy such that the winning probability $p_{ij} \geq 1 - \varepsilon/2$ for every $1 \leq i \leq m$ and $j \in V_i$. But $p_{ij} \geq 1 - \varepsilon/2$ if and only if the winning bias $2p_{ij} - 1 \geq 1 - \varepsilon$, and so the claim follows.

So, in order to prove Theorem 3, we need to construct for every $\varepsilon > 0$, a strategy $(\{Y_{ij}\}, X_j, |\psi\rangle)$ such that $\langle\psi| Y_{ij} \otimes X_j |\psi\rangle \geq 1 - \varepsilon$ for all $1 \leq i \leq m, j \in V_i$. Firstly, what should $|\psi\rangle$ be? It is only natural to take it to be the maximally entangled state on d dimensions - after all, this is what has given us a quantum advantage for a number of other non-local games including the CHSH game and its three player analogue! Furthermore, we already have a handy expression of the winning probabilities from the first problem set: if A and B are any two $d \times d$ matrices, then

$$\langle\psi| A \otimes B |\psi\rangle = \frac{1}{d} \operatorname{tr}(A^T B).$$

Moreover, as both X_j and Y_{ij} must be involutions, we know that

$$2 - \frac{2}{d} \operatorname{tr}(Y_{ij}^T X_j) = \left\| Y_{ij}^T - X_j \right\|^2$$

and so, having $\langle\psi| Y_{ij} \otimes X_j |\psi\rangle \geq 1-\varepsilon$ is the same as ensuring that the distances $\left\| Y_{ij}^T - X_j \right\|$ are all small. Now, remember that we need to connect the representations of Γ to our strategy, so it makes sense first to let the X_js to be the images of the generators x_js of Γ under some representation ϕ and Y_{ij}s to be the images of x_js under another representation ψ_i, and see what constraints they must satisfy. Perhaps this is too much to ask for, but we can, of course, revise our wish-list after making some preliminary observations. From the discussion above, we would obviously like to have

$$\left\| Y_{ij}^T - X_j \right\| = \left\| \psi_i(x_j)^T - \phi(x_j) \right\| \leq O(\varepsilon).$$

To make things notationally simpler, let us instead force Y_{ij} to be the transpose $\psi_i(x_j)^T$ of the image of x_j in some representation, i.e., to have

$$\left\| Y_{ij}^T - X_j \right\| = \left\| \psi_i(x_j) - \phi(x_j) \right\| \leq O(\varepsilon). \tag{2}$$

While for a given i it may be possible to guarantee this for all $j \in V_i$ but one, it is difficult to immediately guarantee this for all $j \in V_i$ simply because we are also forced to satisfy the constraint $\prod_{j \in V_i} Y_{ij} = (-I)^{b_i}$ from Definition 3. So, a natural idea is to let (2) hold for all but one j in V_i and see what constraint that imposes on the remaining j: define W_i to be the set $V_i \setminus \{j_i\}$ where j_i is the maximal index in V_i (just some canonical choice of index for every i), and suppose (2) holds for all $j \in W_i$ and all i. We observe that

$$\|Y_{ij_i}^T - X_{j_i}\| = \left\|(-1)^{b_i}\prod_{j\in W_i}\psi_i(x_j) - \phi(x_{j_i})\right\| \quad \left(\text{we must have } Y_{ij_i} = (-1)^{b_i}\prod_{j\in W_i} Y_{ij} \text{ and commuting } Y_{ij}\right)$$
$$\le \left\|(-1)^{b_i}\prod_{j\in W_i}\phi(x_j) - \phi(x_{j_i})\right\| + \left\|\prod_{j\in W_i}\psi_i(x_j) - \prod_{j\in W_i}\phi(x_j)\right\|$$

How do we bound the second term? Well, we know from (2) that each $\psi_i(x_j)$ is close to $\phi(x_j)$, so it is natural to expect that so are the respective products over $j \in W_i$. We use the following simple bound.

Lemma 1. *If $\{A_j\}_{j\in[r]}$ is a set of commuting self-adjoint unitary matrices and $\{B_j\}_{j\in[r]}$ is a set of unitary matrices with $\|A_j - B_j\| \le \varepsilon$ for each $j \in [r]$, then $\|\prod_{j\in[r]} A_j - \prod_{j\in[r]} B_j\| \le r\varepsilon$.*

Proof Sketch. It suffices to show this for $r = 2$ as otherwise, it follows by simple induction. Note that

$$\|A_1A_2 - B_1B_2\| = \|A_2 - A_1B_1B_2\| \le \|A_2 - B_2\| + \|B_2 - A_1B_1B_2\|$$
$$\le \varepsilon + \|I - A_1B_1\| = \varepsilon + \|A_1 - B_1\| \le 2\varepsilon$$

where we repeatedly used the fact that multiplication by a unitary does not change the (normalized) Hilbert-Schmidt norm. □

So, we have that

$$\|Y_{ij_i}^T - X_{j_i}\| \le \left\|(-1)^{b_i}\prod_{j\in W_i}\phi(x_j) - \phi(x_{j_i})\right\| + |W_i|\cdot O(\varepsilon)$$
$$\le \left\|(-1)^{b_i}\prod_{j\in V_i}\phi(x_j) - I\right\| + O(\varepsilon)$$

where the last equality also uses the fact that $\phi(x_{j_i})^2 = I$, because of the relation $x_{j_i}^2 = e$ in Γ.

Now, if ϕ is actually a representation of Γ (and somehow with $\phi(J) = -I$), then because of the relation $\prod_{j\in V_i} x_j = J^{b_i}$, we would have $(-1)^{b_i}\prod_{j\in V_i}\phi(x_j) - I = 0$. But it is clear from the block of calculations above that we do not actually need such a strong requirement on ϕ, and merely having $\left\|(-1)^{b_i}\prod_{j\in V_i}\phi(x_j) - I\right\| \le O(\varepsilon)$, along with $\phi(J) = -I$ and $\phi(x_j)^2 = I$ for all j would suffice. This is exactly where the concept of ε-representations comes in! So, we have the first item on our wish-list:

$$\text{Given } \varepsilon > 0, \text{ construct an } \varepsilon\text{-representation } \phi \text{ of } \Gamma \text{ such that } \phi(J) = -I \text{ and } \phi(x_j)^2 = I \text{ for all } j. \quad (\dagger)$$

Moreover, note that we need not define the Y representations i.e., ψ_i on i_j: as long as (2) holds for all i and all $j \in W_i$, it suffices to simply let ψ_i be representations of $\mathbb{Z}_2^{W_i}$, let $Y_{ij} := \psi_i(x_j)^T$ and then *define* $Y_{ij_i} = (-1)^{b_i}\prod_{j\in W_i} Y_{ij}$. This way, we have fewer relations to worry about, and we would automatically ensure that $(\{Y_{ij}\}, X_j, |\psi\rangle)$ is indeed a valid tensor product strategy, satisfying the conditions of Definition 3. So given (†) (and in particular, an ε-representation ϕ restricted to the subgroup $\langle x_j : j \in W_i\rangle$ of Γ), we have the second item on our wish-list, assuming we are successful with (†):

Given an ε-representation ϕ of $\mathbb{Z}_2^k$ on a Hilbert space $\mathcal{H}$, construct a representation ψ of $\mathbb{Z}_2^k$ also on $\mathcal{H}$ such that for all $1 \le i \le k$,

$$\|\psi(x_i) - \phi(x_i)\| \le O(\varepsilon) \quad (\ddagger)$$

Let us now focus on (‡) first: starting from the assumption that we have an ε-representation ϕ of $\mathbb{Z}_2^k$. The process of then extracting an ε-representation satisfying (‡) is fortunately a straightforward technique, previously also studied (albeit under a different guise) in [Gle10, FK10].

The idea is as follows. First note that since ϕ is a homomorphism from $\mathbb{Z}_2^k$ to the unitary group $\mathcal{U}(\mathcal{H})$ of a finite dimensional Hilbert space $\mathcal{H}$, $\phi(x_j)$ commute for all generators $x_1, \ldots, x_k$. As a consequence, the set of matrices $\{\phi(x_j)\}$ is simultaneously diagonalizable. Therefore, we may assume that we are working over a basis of $\mathcal{H}$ where each $\phi(x_j)$ is a diagonal matrix. So how do we turn a diagonal matrix whose square is *almost* the identity into one whose square is *exactly* the identity? This is precisely where certain *stability* results, already known in the literature, are useful. We state them below.

Lemma 2. *There is a universal constant C such that for any diagonal matrix X, there is a diagonal matrix D with $D^2 = I$ and $\|D - X\| \leq C \left\|X^2 - I\right\|$.*

Proof Sketch. Taking D where $D_{ii} := \mathrm{sgn}(\mathrm{Re}(X_{ii}))$ for all $1 \leq i \leq d$, where $\mathrm{sgn}(x) := 1$ if $x \geq 0$ and -1 if $x < 0$ works. ☐

Using Lemma 2, we can redefine each $\phi(x_j)$ such that now its square is I. Call this new map ψ_1. But how do we know that ψ_1 is still a $O(\varepsilon)$-representation of $\mathbb{Z}_2^k$? To ensure this, we need a "robustness" result for approximate representations, stated below.

Lemma 3. *Let $G = \langle S : R\rangle$, and let M be the length of the longest relation in R. If ϕ is an ε-representation of G, and ψ is an approximate representation of G with*

$$\|\psi(x) - \phi(x)\| \leq \delta$$

for all $x \in S$, then ψ is an $(M\delta + \varepsilon)$-representation.

So from Lemma 2 and Lemma 3, we have a $O(\varepsilon)$-representation ψ_1 of $\mathbb{Z}_2^k$ that is "close" to ϕ, and such that $\psi_1(x_j)^2 = I$ for all $j = 1, \ldots, k$. Next, we want to modify ψ_1 to arrive at a (exact) representation ψ i.e., we also want all $\psi(x_j)$ to pairwise commute. A natural approach to resolve this task is to proceed sequentially i.e., inductively as follows. We have the following stability result for commuting operators.

Lemma 4. *Suppose $X_1, \ldots, X_n$ are commuting unitary matrices, with $X_i^2 = I$ for all $1 \leq i \leq n$, and Y is a unitary matrix such that $Y^2 = I$ and Y commutes with X_i for all $1 \leq i \leq n-1$. Then there is a universal constant C' and a unitary matrix Z such that $Z^2 = I$, Z commutes with X_i for all $1 \leq i \leq n$, and $\|Z - Y\| \leq C' \|X_n Y - Y X_n\|$.*

Proof Sketch. Taking $Z = \frac{1}{2}(Y + X_n Y X_n)$ works. ☐

Now, apply Lemma 4 with $n = 1$, $X_1 = \psi_1(x_1)$ and $Y = \psi_1(x_j)$ for each $j \geq 2$ to obtain Z_j such that $Z_j^2 = I$ and Z_j commutes with $\psi_1(x_1)$. Define a new $O(\varepsilon)$-representation ψ_2 so that $\psi_2(x_1) := \psi_1(x_1)$, and $\psi_2(x_j) := Z_j$ for all $j \geq 2$. The bound of Lemma 4 guarantees that ψ_2 is $O(\varepsilon)$-close to ψ_1, and furthermore, Lemma 4 is now applicable with $n = 2$, $X_1 = \psi_2(x_1)$, $X_2 = \psi_2(x_2)$, and $Y = \psi_2(x_j)$ for all $j > 2$. We can continue in this manner to obtain a sequence of approximate representations $\psi_1, \psi_2, \ldots, \psi_{k-1}$. It is then straightforward to verify that the final element of this sequence, ψ_{k-1}, is in fact an exact representation that is $O(\varepsilon)$-close to the original ε-representation ϕ!

We have just seen how using these stability ideas help us solve (‡), assuming (†). It turns out that solving (†) is not too hard either, and can be obtained using the same ideas. Recall that we are given that J is non-trivial in approximate representations. In particular, we know that there is a $\delta > 0$ such that for all $\varepsilon > 0$, there is an ε-representation ϕ' with $\|\phi'(J) - I\| > \delta$, and that $\phi'(x_j)^2$ is only *approximately* I for each j. And given ε, we wish to come up with an ε-representation ϕ of Γ such that $\phi(J) = -I$ and $\phi(x_j)^2 = I$ for all j. But we have already seen how to obtain this second condition before, while solving (‡)! And indeed, using Lemmas 2, 3, 4, and the same ideas as in (‡), we can immediately come up with a $O(\varepsilon)$-representation ψ from ϕ' such that

(1) $\psi(x)^2 = I$ for all $x \in \{x_1, \ldots, x_n\ J\}$,

(2) $\psi(J)$ commutes with $\psi(x_j)$ for all $j = 1, \ldots, n$, and

(3) $\|\psi(J) - \phi'(J)\| \leq O(\varepsilon)$.

It now only remains to map J to $-I$. The way that Slofstra does this is rather neat: since $\|\phi(J) - I\| > \delta$, if $\varepsilon < \delta/(2C)$, then

$$\delta < \|\phi'(J) - I\| \leq \|\phi'(J) - \psi(J)\| + \|\psi(J) - I\| \leq \frac{\delta}{2} + \|\psi(J) - I\|,$$

and so $\|\psi(J) - I\| \geq \frac{\delta}{2}$. Therefore, for all $\varepsilon > 0$, there is an ε-representation ψ satisfying conditions (1) and (2), and with $\|\psi(J) - I\| > \frac{\delta}{2}$. Since there is this "gap" between $\psi(J)$ and I, we are able to choose a basis with $\psi(J) = I_{d_0} \oplus (-I_{d_1})$. Since $\psi(x_j)$ commutes with $\psi(J)$ for all $j \in [n]$, we must have $\psi = \psi_0 \oplus \psi_1$, where ψ_a is an approximate representation of dimension d_a, and $\psi_a(J) = (-I)^a$, for $a \in \{0, 1\}$. Since $\psi(x_j)^2 = I$, we also have $\psi_a(x_j)^2 = I$ for all $j \in [n]$ and $a \in \{0, 1\}$. Finally, we can simply choose ϕ to be the *projection* ψ_1 of the $O(\varepsilon)$-representation ψ! The fact that ψ_1 is still a $O(\varepsilon)$-representation of Γ is easy to check, by verifying it separately for each word and relation in Γ.

4 Other Aspects of Non-Local Games and Open Problems

We now present some other related applications of approximate representation theory in non-local games.

4.1 Rigidity and Amenability of Groups

In this subsection, we describe how the rigidity of non-local games can also be explained using approximate representation theory. Informally, for a non-local game G with quantum value $\omega_q^*(G)$, we say that G is rigid if:

- All strategies achieving the optimal quantum value $\omega_q^*(G)$ are equivalent up to local isometries.
- For $\varepsilon > 0$, all strategies achieving winning probability $\omega_q^*(G) - \varepsilon$ are $\delta(\varepsilon)$ close to the optimal strategy in some norm up to local isometries, for some function satisfying $\lim_{\varepsilon \to 0} \delta(\varepsilon) = 0$.

There are numerous games that have this property, including the CHSH game and the Magic Square game that we studied in class. While the rigidity of these games can be proven using some clever linear algebra tricks, it turns out that approximate representation theory, and in particular the Gowers-Hatami theorem, provides a systematic and unifying way to prove rigidity of games. One version of Gowers-Hatami is stated in [Vid18], which we state below.

Definition 8. Given a finite group G, $\varepsilon \geq 0$ and a d-dimensional positive semidefinite matrix σ of trace one, we say that a function $f : G \to \mathcal{U}(\mathbb{C}^d)$ is an (ε, σ)-approximate representation if

$$\mathbb{E}_{x,y \in G} \mathrm{Re}[\mathrm{Tr}[f(x)^* f(y) f(x^{-1}y)\sigma]] \geq 1 - \varepsilon$$

where the expectation is taken over uniformly random elements x, y chosen from the group G.

Note that this notion of approximate representation is different from the notion of approximate representation previously introduced in Definition 6 since here, we have a state σ that could be an arbitrary density matrix. Using this definition, we can state the Gowers-Hatami theorem, which states that these approximate representations of a finite group are indeed close to true representations of the group after some isometry has been applied.

Theorem 4 (Gowers-Hatami). *Given a finite group G, and an (ε, σ)-representation f in $\mathbb{C}^d$, then there exists some $d' \geq d$, an isometry $V : \mathbb{C}^d \to \mathbb{C}^{d'}$ and a representation $g : G \to \mathcal{U}(\mathbb{C}^{d'})$ such that*

$$\mathbb{E}_{x \in G} \, ||f(x) - Vg(x)V^*||_\sigma^2 \leq O(\varepsilon)$$

where $||A||_\sigma^2 = Tr(AA^*\sigma)$ and the expectation is over a uniformly random element $x \in G$.

The Gowers-Hatami theorem can be proven using Fourier analysis. Recall that for functions f over a finite group G, f can be decomposed into Fourier components according to the irreducible representations of G. We will omit the proof here but for details, see [Vid18] or [Gow16]. We will instead concentrate on how the Gowers-Hatami theorem can be applied towards proving the rigidity of non-local games. For instance, suppose that in some game, Alice measures two binary observables A_0, A_1 on $\mathbb{C}^d$ and is somehow able to conclude that there is some state ρ_A for which $||A_0A_1 + A_1A_0||_{\rho_A}^2 \leq O(\varepsilon)$. That is, A_0, A_1 approximately anticommute on the state ρ_A, much like the Pauli observables.

Consider the dihedral group of order 4, the symmetry group of a square, with 8 elements and the presentation $D_4 = \langle r, s | r^4 = s^2 = (sr)^2 = 1 \rangle$, and a map $f : D_4 \to \mathcal{U}(\mathbb{C}^d)$ given by

$$f(r) = A_0A_1 \quad f(r^2) = -I \quad f(r^3) = -A_0A_1 \quad f(r^4) = f(e) = I$$

$$f(s) = A_0 \quad f(rs) = A_1 \quad f(r^2s) = -A_0 \quad f(r^3s) = -A_1$$

The approximate commutation relation between A_0, A_1 and the condition that $A_0^2 = A_1^2 = I$ together imply that f is an (ε, ρ_A)-approximate representation of D_4. Hence, the Gowers-Hatami theorem then implies that there is an isometry that takes f close to an exact representation. Furthermore, the representation theory of D_4 is straightforward, with a unique two-dimensional irreducible representation ρ such that $\rho(r^2) \neq I$. In fact, this representation sends $\rho(r) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $\rho(s) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, and $\rho(rs) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. We can recognize these matrices as Pauli matrices where s is mapped to the Pauli Z and the product rs is mapped to the Pauli X. All other irreducible representations are one-dimensional and send $\rho(r^2) = 1$. Hence, it follows that the representation g to which f is close must contain a 2-dimensional irreducible component and so, we conclude that Alice's operators A_0, A_1 are close to Pauli operators on her "part" of the state, ρ_A. This gives an alternative approach to proving ridigity of the CHSH game and is explored in detail in [Vid18]. Furthermore, the Gowers-Hatami approach is also used to prove the rigidity of the Magic Square Game and an extension of it called the Magic Pentagram Game in [CS19].

The Gowers-Hatami theorem can be extended to a class of (possibly infinite) groups called *amenable groups*. We now define these groups and provide some examples.

Definition 9. Let G be a countable group. We say that G is **amenable** if there is a measure $\mu : \mathcal{P}(G) \to [0, 1]$ assigning each subset a number such that:

1. $\mu(G) = 1$
2. Given finitely many disjoint sets $A_1, \ldots, A_n \subseteq G$, we have $\mu(\bigcup_{i=1}^n A_i) = \sum_{i=1}^n \mu(A_i)$.
3. If $A \subseteq G$ and $g \in G$, we have $\mu(gA) = \mu(A)$ where $gA = \{ga : a \in A\}$.

In other words, G is amenable if there is a finitely additive, left-transitionally invariant probability measure on G. There are numerous equivalent conditions for amenability that can be found in [Tao09].

Example 1. (a) All finite groups are amenable since we can use the counting measure $\mu(A) = \frac{|A|}{|G|}$ for any subset $A \subseteq G$.

(b) All countable Abelian groups and all countable solvable groups are amenable. This is shown in [Gar13, Section 2]. Recall a group is solvable if one can find a chain of groups $1 = G_0 \leq G_1 \leq \cdots \leq G_k = G$ where G_i is a normal subgroup of G_{i+1} and the quotient group G_{i+1}/G_i is Abelian.

(c) In contrast, the free group on two generators $\mathcal{F}_2 = \langle a, b \rangle$ can be shown not to be amenable. This is shown in [Gar13, Section 1]. In fact, the non-amenability of $\mathcal{F}_2$ can be used to explain the famous *Banach-Tarski paradox* that states that a ball in $\mathbb{R}^3$ can be seemingly divided into a finite number of pieces and rearranged to get back two copies of itself but this cannot be done for a circle in $\mathbb{R}^2$. This is because the non-amenable group $\mathcal{F}_2$ embeds into the isometry group of $\mathbb{R}^3$ but not the isometry group of $\mathbb{R}^2$.

Amenability is a useful notion since integration (or expectation) of a function of G is well-defined over an amenable group by using the Lebesgue integral over the measure μ. Furthermore, stability theorems for finite groups like the Gowers-Hatami theorem can be extended to amenable groups. This is the main theorem and technical contribution of [DCOT19].

Theorem 5 ([DCOT19], Theorem 5.2)**.** *Let G be an amenable group and let $\mathcal{M}$ be a von Neumann algebra with (sufficiently nice) norm $||\cdot||$ with unitary group $\mathcal{U}(\mathcal{M})$. If $\phi : G \to \mathcal{U}(\mathcal{M})$ is a map satisfying $\mathbb{E}_{h\in G}||\phi(gh) - \phi(g)\phi(h)|| \leq \varepsilon$ holds for all $g \in G$, then there exists a von Neumann algebra $\mathcal{N}$, a representation $\rho : G \to \mathcal{N}$ and an isometry $U : \mathcal{M} \to \mathcal{N}$ such that*

$$||\phi(g) - U^*\rho(g)U|| \leq O(\epsilon).$$

A von Neumann algebra is a subalgebra of $\mathcal{B}(\mathcal{H})$ (the algebra of bounded operators on a Hilbert space) that is closed in the weak operator topology, a coarser topology that the one induced by the norm. In particular, the finite-dimensional Gowers-Hatami theorem can also be recovered from the proof in the more general setting since all finite-dimensional von Neumann algebras happen to be matrix algebras. We choose not to get into further details because for us, the more important reason for stating above stability theorems for amenable groups is to motivate the following questions that attempt to connect these theorems to the rigidity of non-local games.

Question 1. In [CS19], it is conjectured that the stability theorems for amenable groups in [DCOT19] can be used to prove rigidity guarantees for linear system games where the solution group is infinite. Can this proof indeed be carried out? If so, this could potentially be used to analyze some of the games constructed in [Slo19] that separate the correlation sets $\mathcal{C}_{qa}$ and $\mathcal{C}_{qs}$ to give rigidity guarantees for strategies for non-local games G where an infinite dimensional quantum strategies must be used to play perfectly.

As an intermediate step towards this goal, are there sufficient conditions for when the solution group is solvable or amenable?

We also have the following question about the Gowers-Hatami theorem for finite groups.

Question 2. In [Slo19], it is shown that approximate representations of d for the hypercube $(\mathbb{Z}_2)^k$ are close to exact representations of the **same** dimension d (this is the condition (‡) from previous section). Can we characterize all finite groups where there is no blow-up in dimension in the Gowers-Hatami theorem? If so, are there interesting implications to self-testing and rigidity?

Finally, [DCOT19] asks if there is a converse to their theorem.

Question 3. If a group G satisfies the property that all approximate representations must be close to some exact representation (i.e., a Gowers-Hatami or [DCOT19]-like stability theorem), must G be an amenable group? If true, in the context of linear system games, it means that the amenability of the solution group Γ is a necessary condition for the rigidity of the game.

4.2 Sofic and Hyperlinear Groups

Another interesting notion connected to non-local games is that of sofic and hyperlinear groups. Essentially, a group is sofic if it can be approximated in some sense by finite symmetric groups. Similarly, a group is hyperlinear if it is approximated by finite dimensional unitary groups. In this section we assume that all groups are countable. We use the definitions from [Pes08].

Definition 10. The Hamming metric d_{S_n} on S_n is defined by $d_{S_n}(\sigma_1, \sigma_2) = \frac{1}{n}|\{i : \sigma_1(i) \neq \sigma_2(i)\}|$.

Definition 11. A group G is **sofic** if for every $\varepsilon > 0$ and finite subset $F \subseteq G$, there is an integer n and a map $\Phi : F \to S_n$ such that

1. $d_{S_n}(\Phi(gh), \Phi(g)\Phi(h)) < \varepsilon$ wherever well-defined,
2. $d_{S_n}(\Phi(1_G), e) < \varepsilon$ if $1_G \in F$ where $e \in S_n$ is the identity element,
3. $d_{S_n}(\Phi(g), \Phi(h)) \geq 1/4$ for all $g \neq h$ in F.

The choice of the constant $1/4$ is arbitrary and can be replaced by any constant $0 < c < 1$ for an equivalent definition (see [Pes08, Theorem 3.5]). The definition of a hyperlinear group is similar except with S_n and Hamming distance replaced by the unitary group $\mathcal{U}(\mathbb{C}^n)$ and Hilbert-Schmidt distance.

Definition 12. A group G is **hyperlinear** if for every $\varepsilon > 0$ and finite subset $F \subseteq G$, there is an integer n and a map $\Phi : F \to \mathcal{U}(\mathbb{C}^n)$ such that

1. $||\Phi(gh) - \Phi(g)\Phi(h)|| < \varepsilon$ wherever well-defined,
2. $||\Phi(1_G) - I_d|| < \varepsilon$, where I_d is the d-dimensional identity if $1_G \in F$,
3. $||\Phi(g) - \Phi(h)|| \geq 1/4$ for all $g \neq h$ in F.

Again the choice of the constant $1/4$ is arbitrary so long as it lies in some small range. Note that we could have defined the notions of sofic groups and hyperlinear groups more abstractly by considering embeddings into an ultrapower of symmetric or unitary groups. We have the inclusions that **all amenable groups are sofic** ([PK12, Section 6]), and **all sofic groups are hyperlinear** ([PK12, Collorary 5.3]).

The relevance of these notions to the theory of quantum non-local games follows from the connection with the Connes' embedding conjecture. While we choose not to state its precise statement here, the Connes' embedding conjecture implies that all countable discrete groups are hyperlinear. However, the recent proof of the equality MIP* = RE also disproves the Connes' embedding conjecture and it is now plausible that there are groups that are not hyperlinear. Such a result would also imply that there are groups that are not sofic, which would resolve yet another problem that is open. This motivates the following questions.

Question 4. Can we construct a non-hyperlinear group using linear system games? In particular, Slofstra suggests in [Slo20] that constructing a linear system game with a perfect $\mathcal{C}_{qc}$ strategy but no perfect $\mathcal{C}_{qa}$ strategy would imply that the there is a non-hyperlinear group. This would also give a complete characterization on where there is a difference between perfect finite-dimensional quantum strategies and perfect arbitrary-dimensional quantum strategies for binary constraint system games corresponding to cases where the corresponding constraint satisfaction problem can be solved in polynomial time, following work in [AKS17] that studied operator solutions for constraint satisfaction problems such as 2-SAT.

Question 5. As an intermediate step towards showing that there is a non-hyperlinear group, one could perhaps first show that there is a non-sofic group. Can soficity of a solution group G be characterized in terms of strategies for an associated linear system game? *The well-studied representation theory of the symmetric group may be relevant to this problem.* It is worth noting that the solution group G of the linear system game Slofstra constructs to separate $\mathcal{C}_{qs}$ with $\mathcal{C}_{qa}$ where the element $J \in G$ is non-trivial in approximate finite-dimensional representations but trivial in exact representations is a sofic group.

5 Conclusions

There are rich and profound connections between algebra, functional analysis, representation theory and quantum non-local games. In particular, these relationships have already been utilized to determine when a linear system game has a perfect strategy in various models of quantum mechanics by analyzing its solution group. Furthermore, a better understanding of linear system games and perfect quantum strategies for them seems to be a viable strategy towards resolving the stronger form of the Connes' embedding conjecture, which speculates the existence of a non-hyperlinear group.

There are other areas which warrant further exploration into these connections: these include understanding rigidity of games which allow for infinite dimensional strategies, and extending the solution group framework to analyze a broader class of non-local games, rather than linear system ones. For instance, games that still do not seem to be well-understood from an algebraic point of view include games that are not pseudo-telepathic but still display a quantum advantage, games where answers are not binary-valued, or games where there are more than two players involved. These areas will inevitably be interesting areas of research in the theory of non-local games that can be used to further make progress on related questions in quantum complexity theory and pure mathematics.

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