Quantum Computing Cluster State Model

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1 Introduction

The traditional circuit-based model of quantum computation is strictly based on unitary evolution. Especially in view of the Principle of Deferred Measurement in the context of implementation of algorithms with the circuit model, the role of measurement is secondary to unitary quantum gates - a quantum circuit, sans the final measurements, is the application of a carefully crafted unitary transformation to a carefully chosen initial quantum state. Measurement is a non-unitary operation and thereby ruins quantum states.

The *cluster state model* of quantum computing, also known as a "one-way quantum computer", is based on the idea of performing a sequence of single-qubit measurements on some large initial state, resulting in an output in the form of a several-qubit state. In particular, measurement is essential to the entirety of any computation, in contrast to the circuit model. This model was first introduced in [1] (see also [2]), and it has been studied from both the theoretical and experimental implementation points of view.

The initial state of a computation in this model, called a cluster state, is a specially prepared entangled state associated to a particularly chosen graph. To perform a quantum computation on a cluster state, the rules of the game are: (a) the allowed operations are single-qubit measurements in any basis on any of the nodes of the graph, (b) the basis in which to measure at every stage of the sequence is conditioned on the outcome of the previous measurement, and (c) efficient classical processing is allowed on the measurement outcomes. This model of quantum computing, somewhat surprisingly, is able to simulate any traditional quantum circuit, and thus is a good model of quantum computation.

There are physical implementations of cluster-state quantum computers, and this is an active research area. The traditional linear optics methods, using discrete qubit states, have the disadvantage that the creation of the cluster state desired in a specific computation must be done probabilistically [3]. However, there are implementation schemes which use continuous-variable systems which do not have this property [4], and thus are desirable experimentally. In these schemes, the cluster state is instead created using squeezed optical modes.

We will describe what is a cluster state and how the cluster-state model of quantum computing works. We will then demonstrate why the cluster-state model is universal for quantum computing by explaining how a universal set of quantum gates can be simulated using the framework. Finally, we will discuss practical implementations, and in particular continuous-variable methods.

2 Basics of cluster states

A cluster state is an initial, specially prepared entangled quantum state in a measurement-based quantum computation which we will outline. Such a state is conveniently described as the result of a preparation procedure which depends on a graph. For quantum computation, the graph is chosen based on the particular desired algorithm. In this section, we outline basic definitions and explain the idea behind the model.

2.1 Definition of a cluster state

Let $G = (V, E)$ be a simple graph on n vertices, with $V = \{v_1, \ldots, v_n\}$ and edges $e_{ij} = \{v_i, v_j\} \in E$. An n-qubit *cluster state* on G is a quantum state $|\psi_G\rangle \in (\mathbb{C}^2)^{\otimes n}$ given by

$$|\psi_G\rangle = \prod_{e_{ij} \in E} (\mathrm{CZ})_{ij} |+\rangle^{\otimes n} \tag{1}$$

Here, $(\mathrm{CZ})_{ij}$ denotes the CPHASE gate applied on qubits i, j, which is

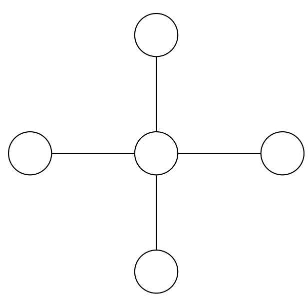
$$\mathrm{CZ} |ab\rangle = (-1)^{ab} |ab\rangle, \quad a, b \in \{0, 1\}$$

Note also that the various $(\mathrm{CZ})_{ij}$ commute with each other (which is a consequence of the definition above), so the product in (1) is well-defined.

In other words, an n-qubit state is obtained by the following procedure:

- Choose a simple graph G on n vertices;
- Prepare n qubits, each in the state $|+\rangle$ (here, a correspondence between qubits and vertices of G is implied);
- Apply a CPHASE gate to each pair of qubits whose corresponding vertices are connected by an edge in G.

For example, consider $|\psi_G\rangle$ associated to the following graph G.



In this case the state $|\psi_G\rangle$ is constructed starting from $|+++++\rangle$ (where the third qubit corresponds to the central node) and applying four CPHASE gates, which results in

$$|\psi_G\rangle = \Big(|++0\rangle + |--1\rangle \Big) |++\rangle = |++\rangle \Big(|0++\rangle + |1--\rangle \Big)$$

Note that this state looks asymmetric despite G being symmetric, but this is only a consequence of presentation (and ultimately it is a consequence of CPHASE being symmetric despite a priori seeming asymmetric).

2.2 An alternate characterization

The definition of a cluster state given above is convenient, but one may object to the fact that two-qubit quantum gates are used in the preparation of a supposedly measurement-only quantum computer. In this subsection, we describe a way to prepare a cluster state which circumvents the application of two-qubit unitaries as in the formal definition, thereby exemplifying a truly measurement-based framework of computation.

Specifically, beginning with the graph-based cluster state $|\psi_G\rangle$, consider the stabilizer group S of $|\psi_G\rangle$. Define, for $v \in V$, the operator

$$S_v = X_v \bigoplus_{v' \in W_v} Z_{v'} \otimes I_{V \setminus W_v}$$

where the subscript indicates which qubit the operator acts on, and $W_v = \{v\} \cup \{v' : \{v, v'\} \in E\}$ is the set of neighbours of v and v itself.

Fact 1. *The cluster state $|\psi_G\rangle$ associated to G satisfies*

$$S_v |\psi_G\rangle = |\psi_G\rangle$$

for all $v \in V$, and it is unique with this property (up to a global phase).

Proof. Choose $v \in V$; without loss of generality, we can assume $v = v_1$ and its neighbours are precisely $v_2, \ldots, v_l$. The last $n - l$ qubits of $|\psi_G\rangle$ are obviously unaffected, so focus on the first l qubits:

$$\begin{aligned} |\varphi\rangle &:= \prod_{j=2}^{l} (\mathrm{CZ})_{1j} |+\rangle^{\otimes l} \\ &= \sum_{k \in \{0,1\}^l} (-1)^{k_1(k_2 + \cdots + k_l)} |k_1 k_2 \cdots k_l\rangle \end{aligned}$$

Now applying S_v, we get

$$\begin{aligned} S_v |\varphi\rangle &= X_1 Z_2 \cdots Z_l \sum_{k \in \{0,1\}^l} (-1)^{k_1(k_2 + \cdots + k_l)} |k_1 k_2 \cdots k_l\rangle \\ &= X_1 \sum_{k \in \{0,1\}^l} (-1)^{(k_1+1)(k_2 + \cdots + k_l)} |k_1 k_2 \cdots k_l\rangle \qquad \text{by definition of } Z \\ &= \sum_{k \in \{0,1\}^l} (-1)^{(k_1+1)(k_2 + \cdots + k_l)} |(k_1 + 1) k_2 \cdots k_l\rangle \\ &= |\varphi\rangle \end{aligned}$$

and therefore $S_v |\psi_G\rangle = |\psi_G\rangle$.

Uniqueness, namely the fact that the $\mathbb{C}$-span of $|\psi_G\rangle$ is the unique line in $\mathcal{H}$ stabilized by the group $\Gamma = \langle S_v \,|v \in V\rangle$, follows from a general theorem (Proposition 10.5 in [5]): if Γ is a subgroup of the Pauli group on n qubits with $n - k$ generators, then the stabilized subspace V_Γ is 2^k-dimensional if (1) the generators S_v are independent, (2) $-I \notin \Gamma$, and (3) Γ is abelian. Point (2) is clear, while point (3) follows because X, Z anticommute, but any pair $S_v, S_{v'}$ acting on the same qubits contains X's and Z'z acting on the same qubits which come in pairs due to the neighbour relation - in particular $S_v S_{v'}$ is -1 raised to an even power times $S_{v'} S_v$. Finally, (1) follows because removing any S_v (say $v = v_i$) from the given generating set introduces a new state

$$|\phi\rangle = |0\rangle^{\otimes(i-1)} \otimes |1\rangle \otimes |0\rangle^{\otimes(n-i)}$$

which is stabilized by the subgroup which is not stabilized by Γ. Points (1)-(3) imply that the stabilized subspace has dimension 2^{n-k}. But in our case, $k = n$, hence the subspace has dimension 1, and we are done. □

Fact 1 means that in particular we can write down a "truly" measurement-only preparation procedure of a cluster state, which involves only measurements and single-qubit unitaries, as follows. We assume given an initial state of $|\psi_0\rangle = |+\rangle^n$, and we first measure all stabilizer generators S_v in this initial state; note this is possible because all S_v commute. The resulting post-measurement state $|\phi\rangle$ is a simultaneous eigenstate of the S_v's. If all eigenvalues are $+1$, then by the fact we must have $|\phi\rangle = |\psi_G\rangle$. Otherwise, there is some subset $W \subset V(G)$ of the vertex set such that $S_v |\phi\rangle = -|\phi\rangle$ for $v \in W$. Then we correct this by applying a sequence of single-qubit operations, namely

$$|\psi_G\rangle = \prod_{v \in W} Z_v |\phi\rangle$$

where Z_v denotes Z applied at the ith qubit, with $v = v_i$ in the conventions of the previous section.

To verify, it suffices to consider the case of a single $v = v_i$ with $S_v |\phi\rangle = -|\phi\rangle$ (so that $|\phi\rangle$ is in the $+1$-eigenspace of all other stabilizer operators). Then first of all, because $ZX = -XZ$,

$$\begin{aligned} S_v Z_i |\phi\rangle &= -Z_i S_v |\phi\rangle \\ &= Z_i |\phi\rangle \end{aligned}$$

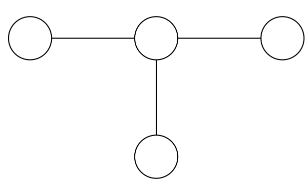
so $Z_i |\phi\rangle$ is now in the $+1$ eigenspace of S_v. On the other hand, since all other $S_{v'}$ are by definition decomposable operators with either the identity or Z acting on the ith qubit, Z_i commutes with $S_{v'}$, and therefore

$$S_{v'} Z_i |\phi\rangle = Z_i S_{v'} |\phi\rangle = Z_i |\phi\rangle$$

so that $Z_i |\phi\rangle$ is still in the $+1$-eigenspace of $S_{v'}$. By Fact 1, we must have $Z_i |\phi\rangle = |\psi_G\rangle$. The proof of the general case is identical. Thus, we have a two-step measurement-based procedure for preparing a cluster state:

- Starting with the $|+\rangle^{\otimes n}$ state, measure all stabilizer generators to obtain a post-measurement state $|\phi\rangle$ which is a simultaneous eigenstate of all generators;
- For every vertex $v = v_i$ such that $S_v |\phi\rangle = -|\phi\rangle$, apply a Z-gate at the ith qubit. The resulting state must be $|\psi_G\rangle$, up to a global phase.

As an example, consider the following graph G:



The stabilizer generators are

$$S_1 = X_1 Z_2,\ S_2 = Z_1 X_2 Z_3 Z_4,\ S_3 = Z_2 X_3,\ S_4 = Z_2 X_4$$

Starting with $|++++\rangle$, measure all four stabilizer generators, and suppose the resulting eigenstate is

$$|\phi\rangle = |+0++\rangle - |-1--\rangle$$

It can be directly verified that $S_2 |\psi\rangle = -|\psi\rangle$, and $S_i |\psi\rangle = |\psi\rangle$ for $i = 1, 3, 4$. Applying Z_2 results in the state

$$Z_2 |\phi\rangle = |+0++\rangle + |-1--\rangle$$

On the other hand, the cluster state for G from the original definition is

$$(\mathrm{CZ})_{12}(\mathrm{CZ})_{23}(\mathrm{CZ})_{24} |++++\rangle = |+0++\rangle + |-1--\rangle$$

exactly as expected.

2.3 Quantum computing with cluster states

The idea of [1] is that these cluster states can be used as a measurement-based model of quantum computing. Specifically, a cluster-state quantum computation proceeds as follows:

- Choose a desirable graph G and prepare $|\psi_G\rangle$,
- Determine a partition of the vertices V into processing nodes V_p and output nodes V_o,
- Assign a partial ordering on V_p; this is the order in which the measurements will be made,
- Assign a measurement basis to each output vertex; the basis may depend on the classical outcome of the *previous* measurements.

As it turns out, this model is universal for quantum computation - any traditional quantum circuit can be simulated by a cluster state computation. We will demonstrate this in the next section, along with some examples. A crucial point is that each of the intermediate sets of measurements is classically controlled by the outcome of the preceeding one. Note also that, as measurement spoils the qubits themselves, a portion of the initial cluster state is destroyed in the procedure, earning it the name “one-way computer”.

3 The cluster state model of quantum computation

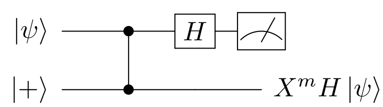
In this section, we will demonstrate the universality of the cluster-state model of quantum computation. Along the way, we will see detailed examples of the computation framework, and in the end we will use these ideas to implement the Deutsch-Josza algorithm.

3.1 Universality of the Cluster State Model

To prove that the cluster state model is universal, it suffices to show that a set of gates which is universal for the circuit model of quantum computing may be replicated by a set of corresponding cluster states. One such universal set for the circuit model of quantum computation is given by the Hadamard gate, the controlled-NOT gate, and the T gate [5], which is a one qubit gate that does the following:

$$T(\alpha|0\rangle + \beta|1\rangle) = \alpha|0\rangle + e^{i\pi/4}\beta|1\rangle$$

We begin with the Hadamard gate H, and the following circuit, known more commonly as one-qubit teleportation [6]:



(2)

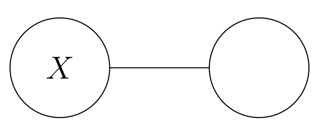
Where m is the output of measuring the first qubit in the computation basis. Letting $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ the state after the application of the Hadamard gate to the first qubit is

$$\alpha|{+}{+}\rangle + \beta|{-}{-}\rangle$$

This can be re-written

$$\begin{aligned}
&\frac{1}{\sqrt{2}}(\alpha(|00\rangle + |01\rangle + |10\rangle + |11\rangle) + \beta(|00\rangle - |01\rangle - |10\rangle + |11\rangle))\\
&= \frac{1}{\sqrt{2}}(|0\rangle \otimes (\alpha|0\rangle + \alpha|1\rangle + \beta|0\rangle - \beta|1\rangle)) + \frac{1}{\sqrt{2}}(|1\rangle \otimes (\alpha|0\rangle - \beta|0\rangle + \alpha|1\rangle + \beta|1\rangle))\\
&= \frac{1}{\sqrt{2}}(|0\rangle \otimes H|\psi\rangle + |1\rangle \otimes XH|\psi\rangle)
\end{aligned}$$

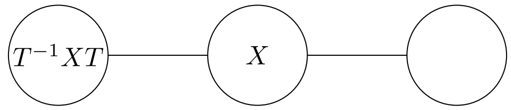
So the output of this circuit is indeed $X^m H|\psi\rangle$. Thus, up to a known Pauli matrix, this circuit applies H to the input qubit $|\psi\rangle$. While this example seems pedantic in the context of quantum circuits (one can simply just apply the Hadamard gate directly to $|\psi\rangle$ in this case), it is useful in the context of cluster-based quantum computation, as it is equivalent to the following cluster-state:



(3)

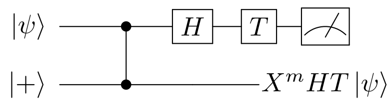
Where $|\psi\rangle$ is the left qubit. This cluster state applies the CPHASE operation to both qubits to entangle them, and then measures the first qubit in the X-eigenbasis. This cluster state is equivalent to the one-bit teleportation circuit because the eigenstates for X are $|+\rangle$ and $|-\rangle$, which means that measuring a state in the X-eigenbasis is equivalent to applying a Hadamard gate to that state and then measuring it in the computational basis.

To implement the T gate as a cluster state, we can consider the following cluster state where the leftmost qubit is our input qubit $|\psi\rangle$ and our rightmost qubit is our output:



(4)

The key here is to note that the columns of the eigenbasis for $T^{-1}XT$ is given by HT; thus, the left two qubits in the cluster state represent the quantum circuit



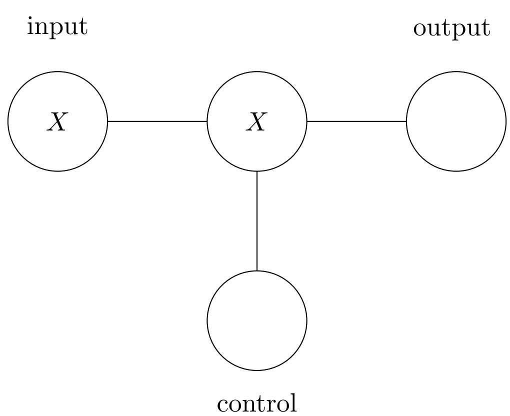
(5)

Where m is 0 or 1 depending on the measurement of the first qubit. To prove that the output of this circuit is as listed, suppose $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$. Notice that after the CPHASE, Hadamard, and T gates have been applied our qubits are in the state

$$\begin{aligned}
&\frac{1}{\sqrt{2}}(|0\rangle \otimes (\alpha\,|0\rangle + \alpha\,|1\rangle + \beta e^{i\pi/4}\,|0\rangle - \beta e^{i\pi/4}\,|1\rangle)) + \frac{1}{\sqrt{2}}(|1\rangle \otimes (\alpha\,|0\rangle - \beta e^{i\pi/4}\,|0\rangle + \alpha\,|1\rangle + \beta e^{i\pi/4}\,|1\rangle)) \\
&= \frac{1}{\sqrt{2}}(|0\rangle \otimes HT\,|\psi\rangle + |1\rangle \otimes X^m HT\,|\psi\rangle),
\end{aligned}$$

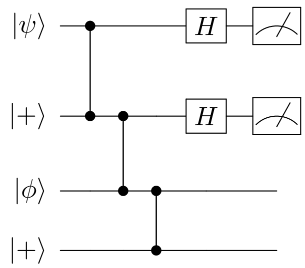
based on the same equations as the one-qubit teleporation case. This output state $X^m HT\,|\psi\rangle$ is then sent through the same quantum circuit as in (2), which gives us a final output state of $X^n H X^m HT\,|\psi\rangle = X^n Z^m T\,|\psi\rangle$, where n is the measurement on the second qubit. Thus, up to a known Pauli matrix, this cluster state implements the T gate.

The final gate that needs to be implemented in order to get a universal gate set for cluster-based quantum computation is the controlled-NOT gate, also known as the cX gate. While 4-qubit and 15-qubit versions of controlled-NOT exist, depending on whether or not we require the control qubit to also be teleported, this paper will focus on the minimal 4-qubit implementation found in [7]. The cluster-state is as follows:



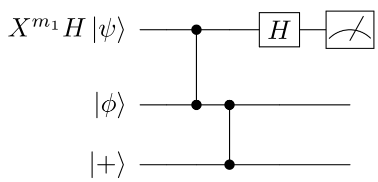
(6)

For the analysis, suppose $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ is our input qubit and $|\phi\rangle = \gamma|0\rangle + \delta|1\rangle$ is our control qubit. The cluster state is equivalent to the following circuit:



(7)

We can process the CPHASE gate between the first and second qubits and the X-measurement of the first qubit before doing anything else. By equation (2) this circuit then simplifies to



Where m_1 is the measurement outcome of the first qubit. We can write $X^{m_1}H|\psi\rangle = (\alpha \pm \beta)|0\rangle + (\alpha \mp \beta)|1\rangle$, which means that the joint state of the system before any operations are applied is

$$\frac{1}{\sqrt{2}}\left((\alpha \pm \beta)|0\rangle + (\alpha \mp \beta)|1\rangle\right) \otimes (|0\rangle + |1\rangle) \otimes (\gamma|0\rangle + \delta|1\rangle)$$

After both $CPHASE$ operations are applied we have the state

$$\frac{1}{\sqrt{2}}((\alpha \pm \beta)\gamma|000\rangle + (\alpha \pm \beta)\delta|001\rangle + (\alpha \pm \beta)\gamma|010\rangle + (\alpha \pm \beta)\delta|011\rangle + (\alpha \mp \beta)\gamma|100\rangle - (\alpha \mp \beta)\delta|101\rangle - (\alpha \mp \beta)\gamma|110\rangle + (\alpha \mp \beta)\delta|111\rangle)$$

After the Hadamard gate is applied we have the state

$$\frac{1}{\sqrt{2}}((\alpha \pm \beta)\gamma|{+00}\rangle + (\alpha \pm \beta)\delta|{+01}\rangle + (\alpha \pm \beta)\gamma|{+10}\rangle + (\alpha \pm \beta)\delta|{+11}\rangle + (\alpha \mp \beta)\gamma|{-00}\rangle - (\alpha \mp \beta)\delta|{-01}\rangle - (\alpha \mp \beta)\gamma|{-10}\rangle + (\alpha \mp \beta)\delta|{-11}\rangle)$$

$$= \frac{1}{\sqrt{2}}(\alpha\gamma|000\rangle \pm \beta\delta|001\rangle \pm \beta\gamma|010\rangle + \alpha\delta|011\rangle \pm \beta\gamma|100\rangle + \alpha\delta|101\rangle + \alpha\gamma|110\rangle \pm \beta\delta|111\rangle)$$

If 0 is the result from measuring the top qubit we get the post-measurement state

$$
\begin{aligned}
\alpha\gamma\left|00\right\rangle \pm \beta\delta\left|01\right\rangle \pm \beta\gamma\left|10\right\rangle + \alpha\delta\left|11\right\rangle &= (\alpha\left|0\right\rangle \pm \beta\left|1\right\rangle) \otimes \gamma\left|0\right\rangle + (\pm\beta\left|0\right\rangle + \alpha\left|1\right\rangle) \otimes \delta\left|1\right\rangle \\
&= CNOT(Z^{m_1}\left|\psi\right\rangle, \left|\phi\right\rangle)
\end{aligned}
$$

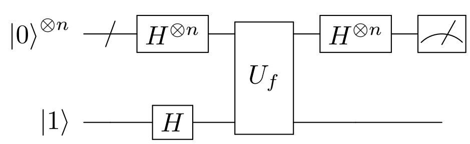
Here the $CNOT(\left|x\right\rangle, \left|y\right\rangle)$ function is assumed to apply the NOT operation on $\left|x\right\rangle$ controlled on $\left|y\right\rangle$. If 1 is the result form measuring the top qubit we get

$$
\begin{aligned}
\pm\beta\gamma\left|00\right\rangle + \alpha\delta\left|01\right\rangle + \alpha\gamma\left|10\right\rangle \pm \beta\delta\left|11\right\rangle) &= (\pm\beta\left|0\right\rangle + \alpha\left|1\right\rangle) \otimes \gamma\left|0\right\rangle + (\alpha\left|0\right\rangle + \pm\beta\left|1\right\rangle) \otimes \delta\left|1\right\rangle \\
&= CNOT(XZ^{m_1}\left|\psi\right\rangle, \left|\phi\right\rangle)
\end{aligned}
$$

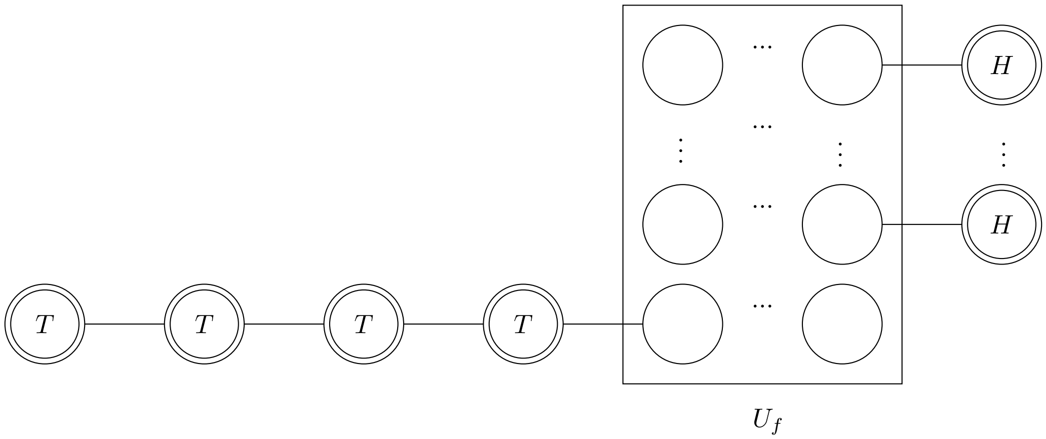
Thus, if m_2 is the value of the second measurement, the final output of this cluster after processing all measurements is $CNOT(X^{m_2}Z^{m_1}\left|\psi\right\rangle, \left|\phi\right\rangle)$. Thus, up to some known Pauli matrix, we have implemented the controlled-NOT gate. This shows the universality of cluster-based quantum computation.

3.2 A cluster-based implementation of the Deutsch-Josza algorithm

As an example of how to utilize the universal operations created in the previous section, we implement the Deutsch-Josza algorithm with cluster states. The Deutsch-Josza algorithm is a common example used to illustrate cases where quantum computation performs exponentially better than classical computation, which is capable of checking whether a given function $f : \{0, 1\}^n \to \{0, 1\}$ is constant or balanced (half the inputs evaluate to $\left|0\right\rangle$ and half the inputs evaluate to $\left|1\right\rangle$), assuming the given function is either constant or balanced. In particular, this algorithm requires only one call to an XOR oracle unitary function U_f, which acts on $n + 1$ qubits and performs the operation $\left|x, b\right\rangle \to \left|x, b \wedge f(x)\right\rangle$, where x is an n-length binary string and b is a single-length binary string. The quantum circuit which implements the Deutsch-Josza algorithm is as follows:



The measurement will return $\left|0\right\rangle^{\otimes n}$ if and only if the function is balanced; a full proof of this fact can be found in [5]. To model such a circuit using a cluster state, we must find an appropriate analogue to the XOR oracle in our cluster-based model. Just as we can assume the existence of a quantum gate representing U_f in the circuit-based model due to the existence of a universal gate set for quantum circuits, so too can we assume the existence of some cluster-state representing U_f. Assuming such a state which takes in $n+1$ qubits as input and outputs $n+1$ qubits, we can consider the following cluster state:



(8)

The box in the above figure represents the XOR oracle in the cluster state model, taking in the qubits in the left of the box as input and outputting the qubits in the right of the box. For brevity's sake, the double-circled nodes represent either the T operator or H operator in the cluster state model, the details of which we covered in the previous subsection. The four T transformations to the left of the oracle modifies the phase of the $(n+1)$st qubit, changing the initial $|+\rangle$ state to a $|-\rangle$ state. This ensures that $|+\rangle^{\otimes n} \otimes |-\rangle$ is the input to the oracle, just as in the original circuit. Finally, the first n output qubits have the Hadamard gate applied to them and are measured, remaining consistent with the circuit.

4 Continuous Variable Cluster States

Preparing a cluster state with stationary qubits such as superconducting qubits and ion-trap qubts is a technologically demanding endeavour due to the number of qubits needed for useful computation [8]. Instead, using qubits based on the degrees of freedom of photons is desirable due to scalability [9]. It has been shown [3] that Bell-state measurements cannot be implemented deterministically using linear optics for polarization entangled photons, motivating the current research in entanglement between the phase quadrature of the modes of an EM field.

A quantum system is called continuous-variable (cv) when its Hilbert space is infinite dimensional and its observables have continuous eigenspectra. The canonical example is the quantization of an N-mode bosonic field, e.g. an electromagnetic field. The Hilbert space for the single mode field is spanned by the Fock basis $\{|n\rangle\}_{n\in\mathcal{N}}$, where n described the number of photons existing in each mode. These basis elements are eigenkets of the number operator $\hat{n} = \hat{a}^\dagger \hat{a}$ where $\hat{a}^\dagger, \hat{a}$ are the creation and annihilation operators. They act on the basis as $\hat{a}^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle$ and $\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle$ and satisfy the commutation relation $[\hat{a}^\dagger, \hat{a}] = 1$. The system can also be described by the phase quadrature operators $\{\hat{p}, \hat{q}\}$ defined as $\hat{q} = \frac{1}{\sqrt{2}}(\hat{a}^\dagger + \hat{a})$ and $\hat{p} = \frac{i}{\sqrt{2}}(\hat{a}^\dagger - \hat{a})$ with corresponding eigenkets $|q\rangle$ and $|p\rangle$ respectively. We can now introduce some important CV gates. Before starting, a good way to think of the new basis $|q\rangle\,, |p\rangle$ is by associating them with the computational basis and Hadamard basis respectively. Instead of switching between these two by a Hadamard gate, a Fourier gate is used and is defined as

$$F = \exp\left(\frac{i\pi}{8}(\hat{q}^2 + \hat{p}^2)\right) \tag{9}$$

In phase-space an arbitrary state can be described by its Wigner quasi-probability distribution or Wigner function. For Gaussian states, eg coherent states, this is completely characterized but the first and second statistical moments of the states. The first moment is defined as $\langle \hat{x} \rangle$ where $\hat{x} = (\hat{q}, \hat{p})$ and the second is the covariance matrix $V_{ij} = \frac{1}{2} \langle \{\Delta\hat{x}_i, \Delta\hat{x}_j\} \rangle$ where $\Delta\hat{x}_i = \hat{x}_i - \langle \hat{x}_i \rangle$ and $\{\}$ is the anti-commutator. For a Gaussian state the Wigner distribution is then defined by

$$W(\hat{x}) = \frac{1}{2\pi \det(V)} \exp\left(-\frac{1}{2}(x - \langle x \rangle)^T V^{-1} (x - \langle x \rangle)\right) \tag{10}$$

For the vacuum state this is

$$W(p, q) = \frac{1}{2\pi} \exp\left(-\frac{1}{2}q^2 - \frac{1}{2}p^2\right) \tag{11}$$

As the co-variance matrix is the identity and the p, q expectation values are zero which is just a zero mean gaussain as should be expected. The Wigner distribution gives a nice way to describe and visualize Gaussian states in phase-space and in the following description of computing one can think of each of the squeezed states as a probability distribution in q, p space.

The CPHASE gate which has been shown to be fundamental for the generation of cluster states is defined as

$$C_Z = \exp\left[\left(\frac{i}{2}\right) \hat{q}_1 \otimes \hat{q}_2\right] \tag{12}$$

In the Heisenberg picture this gate transforms the momentum quadratures as

$$\hat{p}_1 \rightarrow \hat{p}_1 + \hat{q}_2 \qquad \hat{p}_2 \rightarrow \hat{p}_2 + \hat{q}_1 \tag{13}$$

While leaving the position quadratures unchanged. We also define the single mode squeezing operator

$$S(\xi) = \exp(\xi^* \hat{a}\hat{a} - \xi \hat{a}^\dagger \hat{a}^\dagger) \tag{14}$$

Which is used to generate the eignekets of the $\hat{x}$ and $\hat{p}$ operators from coherent beams. Details on this are not included here for brevity and a detailed description can be found in [10]. In a similar way to the earlier sections cluster states are then created by

1.) Preparing highly squeezed vacuum states to approximate momentum eigenstates $|0\rangle_p$ and applying the C_Z gates to connected qumodes.

2.) Performing single-mode measurements on qumodes to decide which basis later measurements will be performed in.

It is useful to introduce the nullifier formalism. Given some state $|\phi\rangle$ the nullifier is the operator H so that $H|\phi\rangle = 0$. For infinitely squeezed vacuum position and momentum eignestates the nullifiers are then $\hat{q}, \hat{p}$ since

$$\hat{p}\,|0\rangle_p = 0 \quad \hat{q}\,|0\rangle_q = 0 \tag{15}$$

Similarly for EPR states the nullifiers are

$$\hat{n}^x_{EPR} = \hat{q}_A - \hat{q}_B \qquad \hat{n}^p_{EPR} = \hat{p}_A + \hat{p}_B \tag{16}$$

Given the simplest possible cluster state of two squeezed momentum eignstates connected by a C_Z gate application using (13) the new nullfiers are then

$$\hat{\delta}_1 = \hat{p}_1 - \hat{q}_2 \qquad \hat{\delta}_2 = \hat{p}_2 - \hat{q}_1 \tag{17}$$

The implementation of such CPHASE gates is challenging. One method is to use two single mode squeezers, beam splitters, and Fourier gates to imitate C_Z.[9][11] To see this we can generate an EPR pair as a two-mode squeezed vacuum (TMSV) by interfering two orthogonally squeezed single mode vacuum states at a 50/50 beam splitter.[12] The nullifiers of this are given by (16). Then we can apply a Fourier gate to one of the modes, corresponding to a $\frac{\pi}{2}$ rotation in phase space i.e.

$$F^\dagger \hat{q} F = \hat{p}, \quad F^\dagger \hat{p} F = -\hat{q} \tag{18}$$

Applying this to the first mode gives the nullifiers

$$\hat{\delta}_1 = \hat{p}_1 - \hat{q}_2 \qquad \hat{\delta}_2 = \hat{p}_2 - \hat{q}_1 \tag{19}$$

Which is the same as the nullifiers obtained from the C_Z gate. This then gives a way to physically implement the C_Z gate in continuous variables and so experimentally create cluster states. As has been stated before to achieve universality two dimensional cluster states are necessary. The example given above does not give an efficient method for creating 2D cluster states. One common method for creating 2D cluster states that has been implemented recently is time-domain multiplexing. [11][9][8] The idea behind this time domain multiplexing is to create a chain of two rails of entangled modes, equivalent to applying C_Z to two qumodes, Then by delaying one of the rails and entangling the the new time-bin corresponding modes the state is curled up to form a 2D cylindrical array of entangled modes.

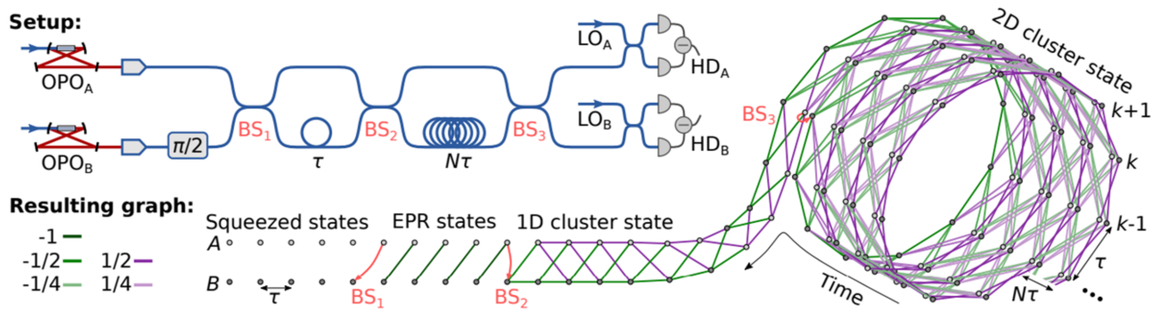


Figure 1: Experimental setup for time-domain multiplexed continuous variable cluster state generation reproduced from Ref [9]. Two Optical Parametric Oscillators (OPOs) generate the squeezed states which are coupled into an optical fibre. The modes are then interfered at beam splitters to create the entangled cylinder as shown above.

Once again, the nullifiers provide a complete description of the state however this time are substantially more complex. They can be derived to be

$$\hat{\delta}^q_k = \hat{q}^A_k + \hat{q}^B_k - \hat{q}^A_{k+1} - \hat{q}^B_{k+1} - \hat{q}^A_{k+N} + \hat{q}^B_{k+N} - \hat{q}^A_{k+N+1} + \hat{q}^B_{k+N+1} \tag{20}$$

$$\hat{\delta}^p_k = \hat{p}^A_k + \hat{p}^B_k + \hat{p}^A_{k+1} + \hat{p}^B_{k+1} - \hat{p}^A_{k+N} + \hat{p}^B_{k+N} + \hat{p}^A_{k+N+1} - \hat{p}^B_{k+N+1} \tag{21}$$

Note this corresponds to the EPR nullifiers and not the nullifiers obtained from applying the C_Z although in a similar fashion to the simple case the C_Z nullifiers can be obtained by applying Fourier gates ($\frac{\pi}{2}$) phase shifts to half of the modes which simply corresponds to a change in the measurement basis. While it is difficult to see the relation exactly the significance is that entanglement is no longer just between two modes but also between their time-bins as can be seen from the recursive nature of (11) and (12) resulting in the 2D topology of the cluster state required for universality.

4.1 Finite Squeezing Induced Noise

In the above description the nodes of the cluster state have been assumed to be perfect momentum eigenstates. That is they are infinitely squeezed. In practice this is un-physical and the finite amount of squeezing leads to added noise and error to the system. For the process of gate teleportation, the output state for some given input $|\psi\rangle = \int dq_1 \psi(q_1) |q_1\rangle$ and ancillary state $|0, V_s\rangle_p$ (where $V_s < 1$ is the variance of the squeezed state) is

$$|\psi'\rangle = \mathcal{M} X(a) F |\psi\rangle \tag{22}$$

Where for the CV case we have replaced X by its CV equivalent and H be the Fourier gate. Note instead of the computation basis and Hadamard basis the CV uses $\hat{q}$ eigenstates and $\hat{p}$ eignstates. Here

$$\mathcal{M} |\psi\rangle \propto \int dq e^{q^2 \frac{V_s}{2}} |q\rangle \langle q|\psi\rangle \tag{23}$$

That is, the distortion is Gaussian with zero mean and variance $\frac{1}{V_s}$[13]. Since any circuit in the cluster state model can be broken down into combinations of these types of circuits, each gate application introduces some Gaussian noise to the state.

In order to combat this and achieve fault tolerance, the use of error correcting codes is required. One of the most standard ways to do this in the CV case is to encode one qubit per oscillator using the GKP code, which protects against phase drift in the encoded qubits, as well as amplitude decay.[14] Universality of quantum computing in the CV cases also requires the use of at least one non-Gaussian operation such as photon number counting, which is popular due to the measurement based nature of the cluster state model. Continuous-variable implementations of the cluster state model with photonic qubits can achieve fault tolerance and universality. Photonic qumodes are advantageous due to their relatively weak interaction with their surrounding environment and room-temperature implementations. With current advancements in generating high quadrature phase squeezing, this approach to quantum computing is becoming increasingly viable.[15]

5 Conclusion

We have introduced the basics of the cluster state model for measurement based quantum computation in both the canonical way, and with a purely measurement based characterization. Its equivalence to the normal model for quantum computation was shown. We then demonstrated the universality of this model as well as an implementation of the Deutsch-Josza with the cluster state model. A physical implementation of this model using continuous variables was then described and some of the complexities involving fault tolerance given. The cluster state model of is interesting model for computation due to its implementations that allow for scalability, universality, and fault tolerance. Recent experimental advances have began to demonstrate these properties making it an interesting area for research both theoretically, and experimentally.

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